The Principle of Strong Mathematical Induction

Let $S(n)$ denote an open mathematical statement (or set thereof) that involves one or more occurrences of the variable n, which represents a positive integer.

Also let $n_0, n_1 \in Z^+$, with $n_0 \leq n_1$

a) If $S(n_0), S(n_0+1), S(n_0+2), \ldots, S(n-1)$, and $S(n)$ are true, and

b) If whenever $S(n_0), S(n_0+1), \ldots, S(K-1)$, and $S(K)$ are true for some (particular but arbitrarily chosen) $K \in Z^+$, where $K \geq n_1$, then the statement $S(K+1)$ is also true;

then $S(n)$ is true for all $n \geq n_0$

Remember: a) is the basis step, &
b) is the inductive step.

Note: n_0 may not be a positive number (depending on the mathematical statement, n_0 might equal $\emptyset$ as in our $S(n) = xy^n$ example).
*It might possibly even be a negative integer (but don't worry about that for this course).

An example

The following calculations demonstrate that it is possible to write the integers 14, 15 & 16 using only 3's and/or 8's as summands:

$$14 = 3 + 3 + 8 \ , \ 15 = 3 + 3 + 3 + 3 + 3 \ ,$$

$$16 = 8 + 8$$

Based on these three results, we make the conjecture:

For every $n \in \mathbb{Z}^+$, where $n \geq 14$

$S(n)$: can be written as a sum of 3's &/or 8's

<u>Proof</u>: From our initial calculations $S(14)$, $S(15)$, & $S(16)$ are true. These establish our basis step, where $n_0 = 14$ & $n_1 = 16$

For the inductive step, we assume the truth of the statements:

$$S(14), S(15), \ldots, S(K-2), S(K-1), \text{ \& } S(K),$$

for some $K \in \mathbb{Z}^+$, where $K \geq 16$

Now if $n = K+1$, then $n \geq 17$ and

$$K+1 = (K-2) + \underline{\underline{3}}$$

Since $14 \leq K-2 \leq K$, from the truth of $S(K-2)$ we know that $S(K-2)$ can be written as a sum of 3's and/or 8's

so $S(K+1) = S(K-2) + \underline{\underline{3}}$ can also be written in this form

Consequently, $S(n)$ is true for all $n \geq 14$

$f(n)$ is said to have an explicit formula iff $f(n)$ can be determined from n alone. (example $f(n) = 2*n$)

Sometimes it is difficult to define a mathematical concept in an explicit manner

Instead it may be easier to define a concept in terms of prior results. In other words

. . .

$$g(n) = h(g(a_1), g(a_2) \ldots, g(a_x)),$$

where $a_1, a_2, \ldots, a_x \in \mathbb{Z}^+$ and $a_1 < a_2 < \ldots a_x < n$
& where the minimum number of prior values needed to compute $g(n)$ is $g(a_1)$

In cases such as these, $g(n)$ has a recursive definition.

This means that while we do not have an explicit formula for $g(n)$ to define the $g(0), g(1), g(2)$, etc sequence, we do have a way of defining $g(n)$ using recursion.

For example, let us consider the integer sequence $a_0, a_1, a_2, a_3, \ldots$ where

$$a_0 = 1, \; a_1 = 2, \; a_2 = 3, \text{ and}$$

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \text{ for all } n \in \mathbb{Z}^+, \text{ where } n \geq 3$$

Here we do not have an explicit formula that defines each a_n in terms of n for all $n \in \mathbb{Z}^+$

For example, if we want to determine the value of a_6, we need to know the values of a_5, a_4, & a_3 (which require us to know the values of a_2, a_1, & a_0) as:

$$a_6 = a_5 + a_4 + a_3$$

$$= [\overbrace{a_4 + a_3 + a_2}^{A}] + [\overbrace{a_3 + a_2 + a_1}^{B}] + [\overbrace{a_2 + a_1 + a_0}^{C}]$$

$$= [(a_3 + a_2 + a_1) + (a_2 + a_1 + a_0) + a_2] + [(a_2 + a_1 + a_0) + a_2 + a_1] + [\underbrace{a_2 + a_1 + a_0}_{C}]$$

$$= [([a_2 + a_1 + a_0] + a_2 + a_1) + (a_2 + a_1 + a_0) + a_2] + [(a_2 + a_1 + a_0) + a_2 + a_1] + [a_2 + a_1 + a_0]$$

$$= [([3+2+1] + 3 + 2) + (3+2+1) + 3] + [(3+2+1) + 3 + 2] + [3 + 2 + 1]$$

$$= 37$$

This solution used to obtain the value of a_6 relies on recursively obtaining the solutions to prior values.

Hint: This recursive equation could be calculated using a recursive program. Try doing this during the tutorial at the end of class (Problem 1)

As we discussed during our lecture on recursive function design, any problem you solve recursively can also be solved iteratively.

Recalling our integer sequence description $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for all $n \in \mathbb{Z}^+$, where $n \geq 3$:
$a_0 = 1$, $a_1 = 2$, & $a_2 = 3$

We can iteratively calculate a_6 as follows:

$$a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6$$
$$a_4 = a_3 + a_2 + a_1 = 6 + 3 + 2 = \underline{11}$$
$$a_5 = a_4 + a_3 + a_2 = \underline{11} + 6 + 3 = 20$$
$$a_6 = a_5 + a_4 + a_3 = 20 + \underline{11} + 6 = 37$$

Problem 2: Create an iterative version of a function for this same number sequence with the same function declaration as the recusive version (Problem 1)

Another example:

We know that $p_1 \wedge (p_2 \wedge p_3) \Leftrightarrow (p_1 \wedge p_2) \wedge p_3$
$\Leftrightarrow p_1 \wedge p_2 \wedge p_3$
where p_1, p_2, & p_3 are any logical statements

How do we extend this for expressions such as $p_1 \wedge p_2 \wedge p_3 \wedge p_4$, where $p_1, p_2, p_3,$ & p_4 are statements.

Sol^n: We introduce the following recursive definition, wherein the concept at the [(n+1)st] stage is developed from the comparable concept at a previous [nth] stage.

Given any statements $p_1, p_2, \ldots, p_n, p_{n+1}$, we define

1) the conjunction of p_1, p_2 by $p_1 \wedge p_2$

2) the conjunction of $p_1, p_2, \ldots, p_n, p_{n+1}$ for $n \geq 2$ by

(LHS)
$$p_1 \wedge p_2 \wedge \ldots \wedge p_n \wedge p_{n+1} \Leftrightarrow$$
(RHS)
$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \wedge p_{n+1}$$

The result in 1) establishes the base for the recursion, while the logical equivalence in 2) provides the recursive process. [Note: the statement on the right-hand side (RHS) of the logical equivalence in 2) is the conjunction of two statements: p_{n+1} & the previously determined statement $(p_1 \wedge p_2 \wedge \ldots \wedge p_n)$.]

Hint: think of 1)

Therefore, we define the conjunction of p_1, p_2, p_3, & p_4 by

$$p_1 \wedge p_2 \wedge p_3 \wedge p_4 \Leftrightarrow (p_1 \wedge p_2 \wedge p_3) \wedge p_4$$

By the associative law of $\wedge$, we find that

$$\begin{aligned}
(p_1 \wedge p_2 \wedge p_3) \wedge p_4 &\Leftrightarrow ([p_1 \wedge p_2] \wedge p_3) \wedge p_4 \\
&\Leftrightarrow [p_1 \wedge p_2] \wedge p_3 \wedge p_4 \\
&\Leftrightarrow (p_1 \wedge p_2) \wedge (p_3 \wedge p_4) \\
&\Leftrightarrow p_1 \wedge p_2 \wedge (p_3 \wedge p_4) \\
&\Leftrightarrow p_1 \wedge [p_2 \wedge (p_3 \wedge p_4)] \\
&\Leftrightarrow p_1 \wedge [p_2 \wedge p_3 \wedge p_4]
\end{aligned}$$

Using the above definition, we can extend these results to the "Generalized Associative Law for $\wedge$" using mathematical induction to prove it.

Let $n \in \mathbb{Z}^+$ where $n \geq 3$, & let $r \in \mathbb{Z}^+$ with $1 \leq r < n$
Then

$S(n)$: For any statements $p_1, p_2, \ldots, p_r, p_{r+1}, \ldots p_n$

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_n) \Leftrightarrow$$

$$p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_n$$

Proof: The truth of the statement $S(3)$ follows from the associative law for $\wedge$ (you would have seen this defined in ensc 252, but you can look it up if need be). This establishes the basis step for our inductive proof. For the inductive step, we assume that $S(k)$ is true for some $k \geq 3$ and all $1 \leq r < k$.

That is we assume the truth of:

$$S(k): (p_1 \wedge p_2 \wedge \ldots \wedge p_r) \wedge (p_{r+1} \wedge \ldots \wedge p_k) \Leftrightarrow$$

$$p_1 \wedge p_2 \wedge \ldots \wedge p_r \wedge p_{r+1} \wedge \ldots \wedge p_k$$

Then we show that $S(k) \Rightarrow S(k+1)$.
When we consider $k+1$ statements, then we must account for all $1 \leq r < k+1$

CASE 1) If $r=k$, then from our recursive definition

$$(p_1 \wedge p_2 \wedge \dots \wedge p_k) \wedge p_{k+1} \Leftrightarrow$$

$$p_1 \wedge p_2 \wedge \dots \wedge p_k \wedge p_{k+1}$$

CASE 2) For $1 \le r < k$, we have

$$(p_1 \wedge p_2 \wedge \dots \wedge p_r) \wedge (p_{r+1} \wedge \dots \wedge p_k \wedge p_{k+1})$$

$$\Leftrightarrow \underline{(p_1 \wedge p_2 \wedge \dots \wedge p_r)} \wedge (\underline{[p_{r+1} \wedge \dots \wedge p_k]} \wedge \underline{p_{k+1}})$$ from Law of Associativity for 3 statements

(3 statements)

$$\Leftrightarrow [\underline{(p_1 \wedge p_2 \wedge \dots \wedge p_r)} \wedge \underline{[p_{r+1} \wedge \dots \wedge p_k]}] \wedge p_{k+1}$$

(2 statements)

$$\Leftrightarrow \underline{[p_1 \wedge p_2 \wedge \dots \wedge p_r \wedge p_{r+1} \wedge \dots \wedge p_k]} \wedge \underline{p_{k+1}}$$

(2 statements)

$$\Leftrightarrow p_1 \wedge p_2 \wedge \dots \wedge p_r \wedge p_{r+1} \wedge \dots \wedge p_k \wedge p_{k+1}$$

Thus it follows by the Principle of Mathematical Induction that the opening statement for $S(n)$ is true for all $n \in \mathbb{Z}^+$, where $n \ge 3$

Questions: Is the above proof done using strong induction or weak induction (justify your answer)

Questions (cont'd):

Apply the same methodology to obtain the

a) "Generalized Associative Law for $\cup$" &

b) "Generalized Associative Law for $\cap$"

You can also recursively define the sum (& product) of n real numbers, where $n \in \mathbb{Z}^+$ & $n \geq 2$.

You can use these definitions to obtain the generalized associative laws for the addition & multiplication of real numbers. ** Hint: this is another opportunity to practice if you need to. **