Note Title 11/15/2007

Lecture - 32 + 33

apply (1) Eqns. of motion

(2) Linear + ang. acc. Eqns. } for a rigid body

to a manipulator arms

to derive the dynamic eqns.

Set of Eqns:

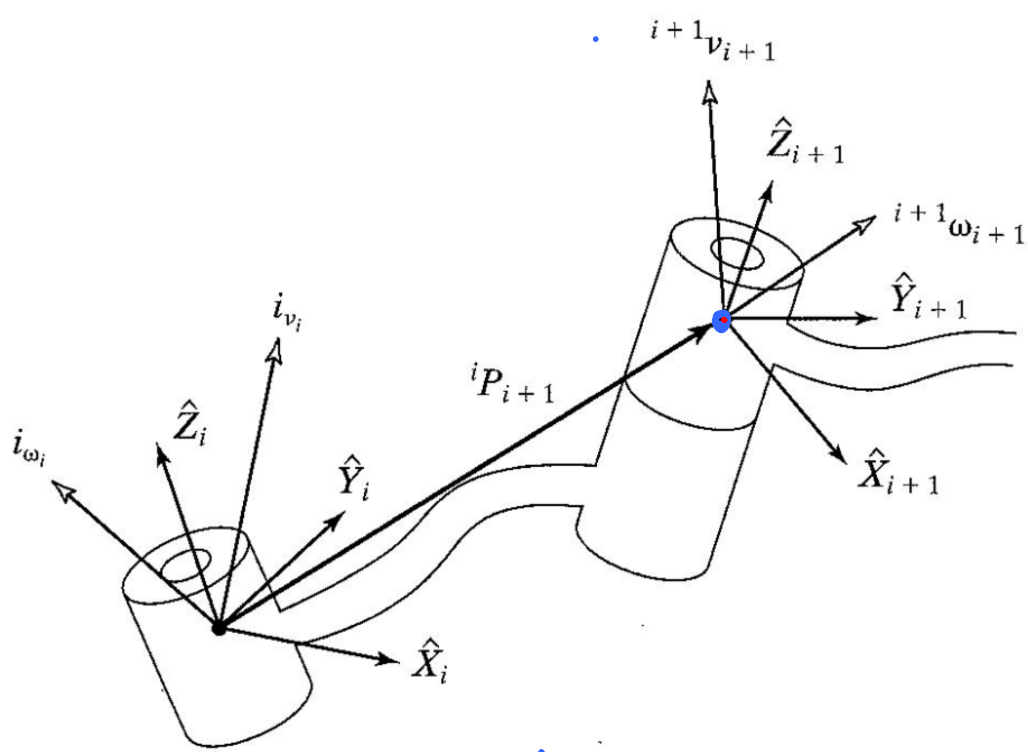
$$ {}^{A}\dot{\Omega}_C = {}^{A}\dot{\Omega}_B + {}^{A}_{B}R\,{}^{B}\dot{\Omega}_C + {}^{A}\Omega_B \times \left({}^{A}_{B}R\,{}^{B}\Omega_C\right) $$

for rev. joint

$$ {}^{A}\dot{V}_Q = {}^{A}\dot{V}_{BORG} + \cancel{{}^{A}_{B}R\,{}^{B}\dot{V}_Q} + \cancel{2\,{}^{A}\Omega_B \times \left({}^{A}_{B}R\,{}^{B}V_Q\right)} + {}^{A}\dot{\Omega}_B \times \left({}^{A}_{B}R\,{}^{B}Q\right) + {}^{A}\Omega_B \times \left({}^{A}\Omega_B \times {}^{A}_{B}R\,{}^{B}Q\right) $$

$$ F = m \cdot \dot{v}_C $$

$$ N = {}^{C}I\,\dot{\omega} + \omega \times {}^{C}I\,\omega $$



and acc.

FIGURE 5.7: Velocity vectors of neighboring links.

[2] Remember that linear velocity is associated with a point, but angular velocity is associated with a

5) <u>Ang. acc.</u> :

$\{A\} = \{0\}$

$\{B\} = \{i\}$

$\{C\} = \{i+1\}$

$$\left.\begin{aligned} \dot{\omega}_i &= {}^0\left({}^0\dot{\Omega}_i\right) \\ {}^i\dot{\omega}_i &= {}^i_0R\,\dot{\omega}_i \end{aligned}\right\} \text{same for } \omega_i$$

$${}^i\dot{\Omega}_{i+1} = {}^i_{i+1}R \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_{i+1} \end{pmatrix}$$

$$= {}^i_{i+1}R\,\ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$\dot{\omega}_{i+1} = \dot{\omega}_i + {}^0_iR\left[{}^i_{i+1}R\,\ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\right]$$

$$+ \omega_i \times \left[{}^{0}_{i}R \; {}^{i}_{i+1}R \; \dot{\theta}_{i+1} \; {}^{i+1}\hat{Z}_{i+1} \right]$$

$${}^{i+1}\dot{\omega}_{i+1} = {}^{i+1}_{0}R \; \dot{\omega}_{i+1}$$

$$R(\vec{P} \times \vec{Q})$$

$$= R\vec{P} \times R\vec{Q}$$

for a rev. joint

$$\boxed{= {}^{i+1}_{i}R \; {}^{i}\dot{\omega}_i + \left[{}^{i+1}_{i}R \; {}^{i}\omega_i \right] \times \dot{\theta}_{i+1} \; {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1} \; {}^{i+1}\hat{Z}_{i+1}}$$

for prism: see text. $\ddot{\theta}_{i+1} = 0$, $\dot{\theta}_{i+1} = 0$

Linear Acc.:

$$\{A\} = \{0\}$$

$$\{B\} = \{i\}$$

$$^{B}Q = {}^{i}P_{i+1}$$

Rev. joint:

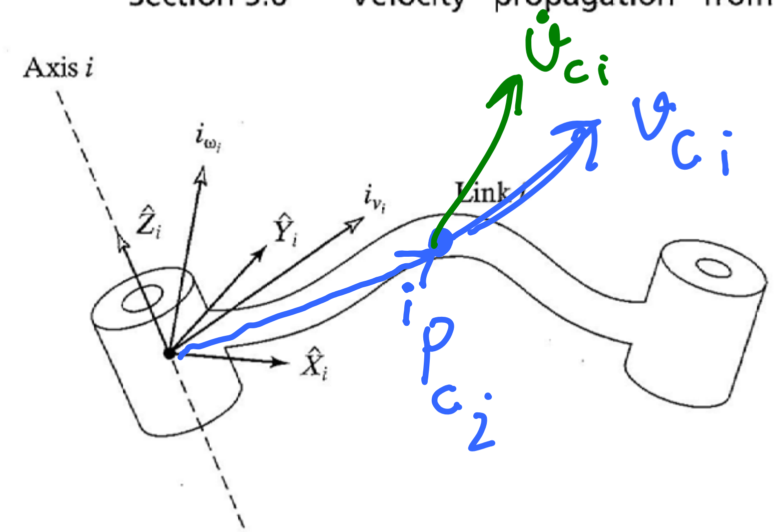
$$\begin{cases} {}^{B}\dot{V}_{Q} = {}^{i}\dot{V}_{i+1} = 0 \\ {}^{B}V_{Q} \quad {}^{i}V_{i+1} = 0 \end{cases}$$

$$\dot{v}_{i+1} = \dot{v}_{i} + \dot{\omega}_{i} \times \left[{}^{0}_{i}R \; {}^{i}P_{i+1} \right]$$

$$+ \omega_i \times \left(\omega_i \times {}^{0}_{i}R\, {}^{i}P_{i+1}\right)$$

$$
\begin{aligned}
{}^{i+1}\dot{v}_{i+1} &= {}^{i+1}_{0}R\, \dot{v}_{i+1} \\
&= {}^{i+1}_{i}R \left[{}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{i+1} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{i+1} \right) \right]
\end{aligned}
$$

we need $\dot{v}_{c_i}$ for N-E eqns. of motion



$P_{c_i} = Q$

$\{B\} = \{i\}$

$\{A\} = \{0\}$

$$\dot{v}_{c_2} = \dot{v}_i + \dot{\omega}_i \times \left[{}^{0}_{i}R \, {}^{i}P_{c_i} \right] + \omega_i \times \left[\omega_i \times {}^{0}_{i}R \, {}^{i}P_{c_i} \right]$$

$$ {}^{i}\dot{v}_{c_i} = {}^{i}_{0}R\,\dot{v}_{c_i} $$

$$ = {}^{i}\dot{v}_i + {}^{i}\dot{\omega}_i \times {}^{i}P_{c_i} + {}^{i}\omega_i \times \left({}^{i}\omega_i \times {}^{i}P_{c_i} \right) $$

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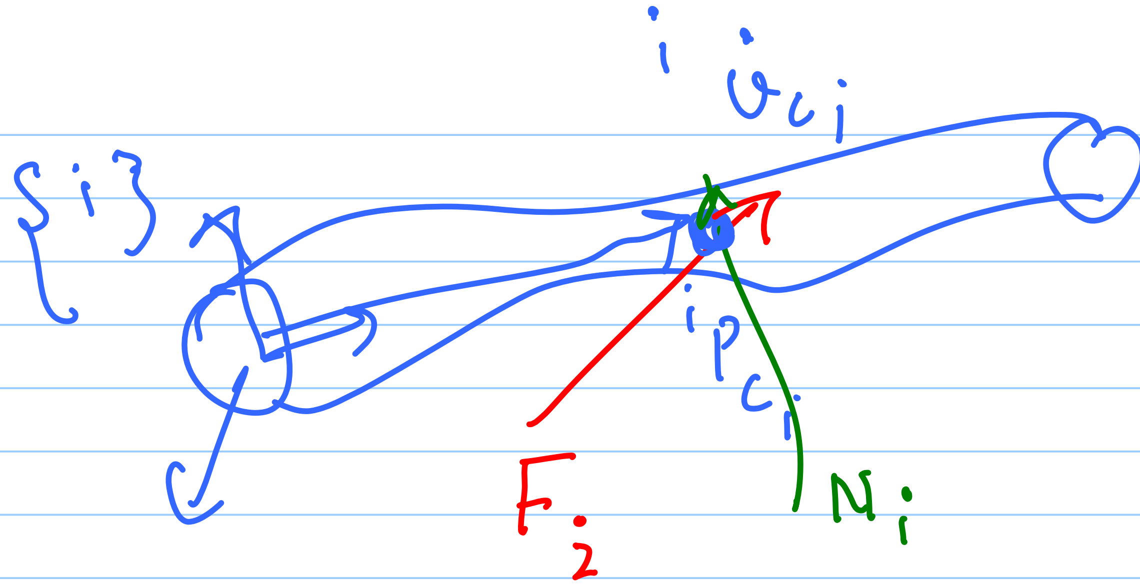
having computed ω_i, $\dot{\omega}_i$, $\dot{v}_{c_i}$, we can apply N-E eqns. of motion to link i to get net force/moment of the centre of mass of link i



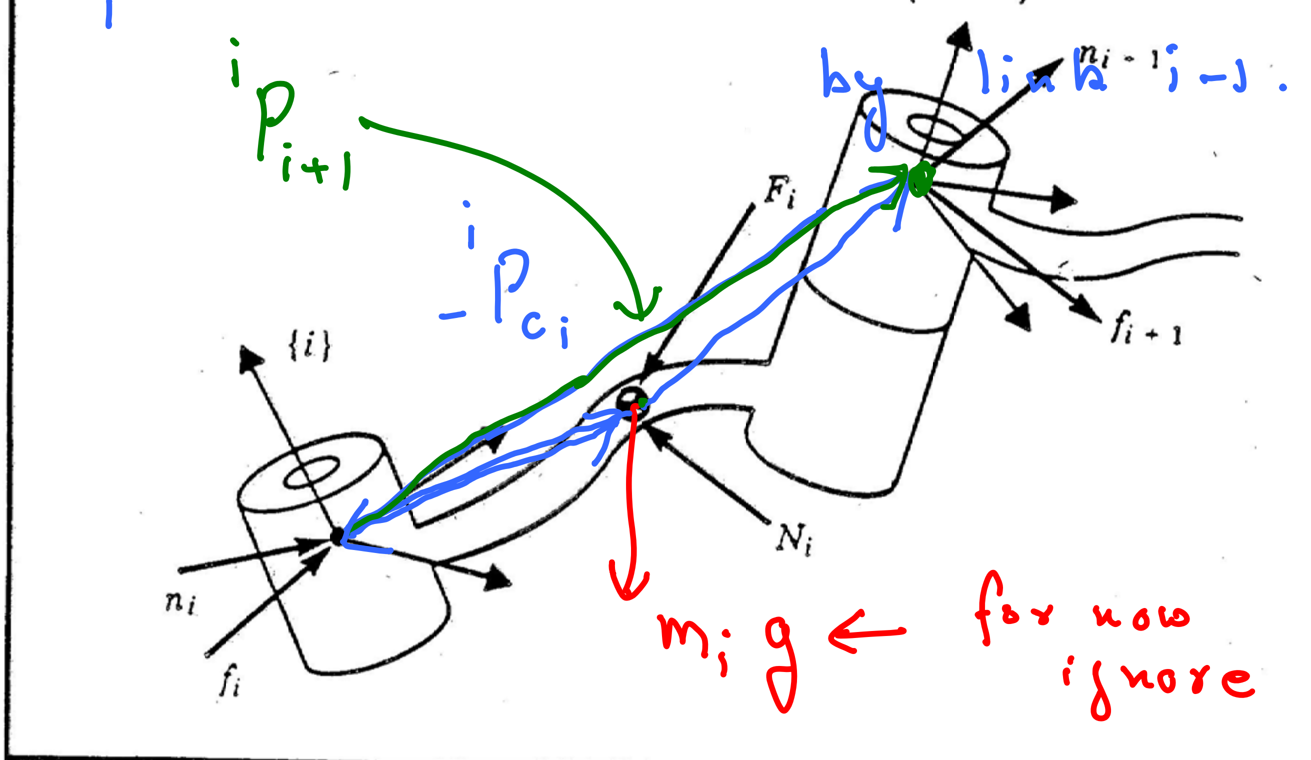
${}^{c_i}I_i$, m_i $\leftarrow$ given

$$F_i = m_i \, {}^{i}\dot{v}_{c_i}$$

$$N_i = {}^{c_i}I_i \, {}^{i}\dot{\omega}_i + {}^{i}\omega_i \times \left({}^{c_i}I_i \, {}^{i}\omega_i \right)$$

f_i = force exerted on link i by link $i-1$.

n_i = moment exerted on link i by link $i-1$.



force / moment balance eqns: ${}^{i}f_i$

force balance

$${}^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}$$

$$\boxed{{}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i}$$

moment balance

$${}^{i}N_i = {}^{i}n_i - {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + \left(-{}^{i}P_{C_i} \times {}^{i}f_i\right)$$

$$+ \left({}^{i}P_{i+1} - {}^{i}P_{C_i} \right) \times {}^{i}_{i+1}R \left(- {}^{i+1}f_{i+1} \right)$$

$$\boxed{\begin{aligned} {}^{i}n_i &= {}^{i}N_i + {}^{i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i \\ &\quad + {}^{i}P_{i+1} \times \left({}^{i}_{i+1}R \, {}^{i+1}f_{i+1} \right) \end{aligned}}$$

$${}^{N+1}f_{N+1} = 0$$

$${}^{N+1}n_{N+1} = 0$$

$\tau_i \rightarrow$ torque (moment ~~axis~~ in $\hat{z}_i$)

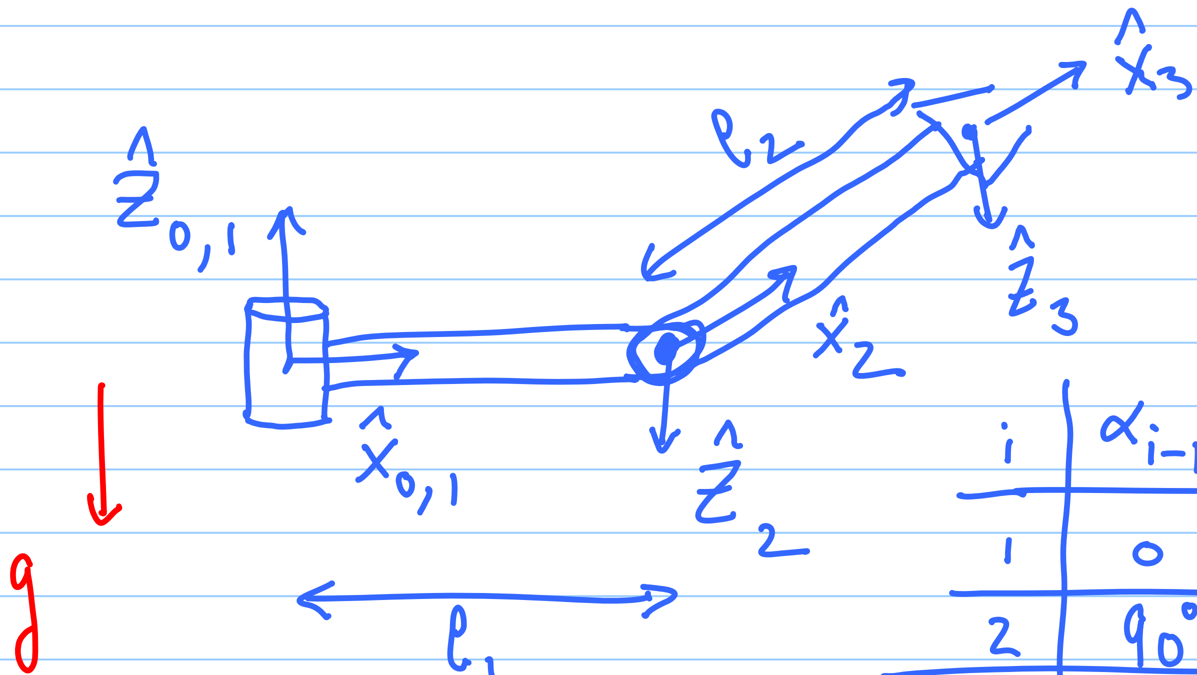
$= \hat{z}$ comp. of ${}^{i}n_i$

$$\tau_i = \left({}^{i}n_i\right) \cdot \left({}^{i}\hat{z}_i\right) \quad \text{rev.}$$

$$f_i = \left({}^{i}f_i\right) \cdot \left({}^{i}\hat{z}_i\right)$$

Planar Case in text.

Dynamic Eqns. for a non-planar R-R manipulator



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	90°	ℓ_1	0	θ_2
3	0	ℓ_2	0	0

$$ {}^{0}_{1}T = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad {}^{1}_{2}T = \begin{pmatrix} c_2 & -s_2 & 0 & l_1 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{2}_{3}T = \left(\begin{array}{c|c} I & \begin{matrix} l_2 \\ 0 \\ 0 \end{matrix} \\ \hline \underline{0} & 1 \end{array}\right) \qquad {}^{1}P_{c_1} = \begin{pmatrix} l_1 \\ 0 \\ 0 \end{pmatrix} \qquad {}^{2}P_{c_2} = \begin{pmatrix} l_2 \\ 0 \\ 0 \end{pmatrix} $$

$$ {}^{c_1}\underline{\underline{I}}_1 = \underline{0} \qquad {}^{c_2}\underline{\underline{I}}_2 = \underline{0} \qquad {}^{0}P_1 = \underline{0} $$

m_1 = mass of link 1

m_2 = " " link 2

forw. it.

linear/ang. vel : ${}^{0}v_0 = 0 \quad \cancel{\boxed{=}} \quad {}^{0}\dot{v}_0 = \begin{pmatrix} 0 \\ 0 \\ 9.8 \end{pmatrix}$

\+ acc.

$${}^{0}\omega_0 = 0 = {}^{0}\dot{\omega}_0$$

$${}^{1}\omega_1 = \cancel{{}^{0}_{1}R \, {}^{0}\omega_0} + \dot{\theta}_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix}$$

$$ {}^{1}\dot{\omega}_1 = {}^{1}_{0}R\,{}^{0}\dot{\omega}_0 + \cancel{{}^{1}_{0}R\,{}^{0}\omega_0 \times} \left(\dot{\theta}_1\,{}^{1}\hat{Z}_1\right) + \ddot{\theta}_1\,{}^{1}\hat{Z}_1 $$

$$ = \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} $$

$$ {}^{1}\dot{v}_1 = \begin{pmatrix} 0 \\ 0 \\ \cancel{9.8}\,g \end{pmatrix} $$

$$ {}^{1}\dot{v}_{C_1} = {}^{1}\dot{\omega}_1 \times {}^{1}P_{C_1} + {}^{1}\omega_1 \times \left({}^{1}\omega_1 \times {}^{1}P_{C_1}\right) $$

$$+\ {}^{1}\dot{v}_{1}$$

$$= \begin{pmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} \ell_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \left[\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} \ell_1 \\ 0 \\ 0 \end{pmatrix} \right]$$

$$+ \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$$

$$= \begin{pmatrix} -\ell_1 \dot{\theta}_1^{2} \\ \ell_1 \ddot{\theta}_1 \\ g \end{pmatrix} \begin{matrix} \rightarrow \text{centripetal} \\ \rightarrow \text{due to ang. acc.} \\ \rightarrow \text{gravity} \end{matrix}$$

Link 2

$${}^{2}\omega_2 = \begin{pmatrix} s_2 \dot\theta_1 \\ c_2 \dot\theta_1 \\ \dot\theta_2 \end{pmatrix}$$

$${}^{2}\dot\omega_2 = \begin{pmatrix} s_2 \ddot\theta_1 + c_2 \dot\theta_1 \dot\theta_2 \\ c_2 \ddot\theta_1 - s_2 \dot\theta_1 \dot\theta_2 \\ \ddot\theta_2 \end{pmatrix}$$

$${}^{2}\dot v_2 = \begin{pmatrix} -l_1 c_2 \dot\theta_1^2 + s_2 g \\ l_1 s_2 \dot\theta_1^2 + c_2 g \\ -l_1 \ddot\theta_1 \end{pmatrix}$$

$$
{}^{2}\dot{v}_{c_2} = \begin{pmatrix} -(l_1 c_2 + l_2 c_2^2)\dot{\theta}_1^2 - l_2 \dot{\theta}_2^2 + s_2 g \\ (l_1 s_2 + l_2 s_2 c_2)\dot{\theta}_1^2 + l_2 \ddot{\theta}_2 + c_2 g \\ 2 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 - l_1 \ddot{\theta}_1 + l_2 c_2 \ddot{\theta}_1 \end{pmatrix}
$$

$$
{}^{1}F_1 = \begin{pmatrix} -m_1 l_1 \dot{\theta}_1^2 \\ m_1 l_1 \ddot{\theta}_1 \\ m_1 g \end{pmatrix} = m_1 {}^{1}\dot{v}_{c_1}
$$

$$
{}^{1}N_1 = \underline{0}
$$

N-E eqn.s of motion

$${}^{2}F_2 = m_2 \, {}^{2}\dot{v}_{c_2}$$

$${}^{2}N_2 = \underline{0}$$

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backward iter.

$${}^{3}f_3 = 0 \qquad {}^{3}n_3 = 0$$

<u>link 2</u> :

$${}^{2}f_2 = {}^{2}F_2$$

$$ {}^{2}n_2 = {}^{2}P_{c_2} \times {}^{2}\bar{F}_2 $$

$$ = \begin{pmatrix} 0 \\ \times \\ \times\times. \end{pmatrix} $$

$$ \tau_2 = \times\times = m_2 \ell_2 (\ell_1 + \ell_2 c_2) s_2 \dot\theta_1^2 - m_2 g_2 \ell_2 c_2 $$

$$ = \qquad + m_2 \ell_2^2 \ddot\theta_2 $$

$$ \times = m_2 \ell_1 \ell_2 \ddot\theta_1 - 2 m_2 \ell_2^2 s_2 \dot\theta_1 \dot\theta_2 $$

$$+ \, m_2 l_2^2 \, \ddot{\theta}_1$$

Link 1:

$$^{1}f_1 = {}^{1}_{2}R \, {}^{2}f_2 + {}^{1}F_1$$

$$^{1}n_1 = \cancel{{}^{1}N_1} + {}^{1}_{2}R \, {}^{2}n_2 + {}^{1}P_{c_1} \times {}^{1}F_1 + {}^{1}_{2}R \, {}^{2}f_2$$

$$ {}^{1}n_1 = \begin{pmatrix} x \\ xx \\ xxx \end{pmatrix} $$

$$ \tau_1 = xxx = \left(m_1 \ell_1^2 + m_2 \ell_1^2 + m_2 \ell_2^2 c_2 + 2 m_2 \ell_1 \ell_2 c_2\right) \ddot{\theta}_1 - 2\left(\ell_1 + \ell_2 c_2\right) m_2 \ell_2 s_2 \dot{\theta}_1 \dot{\theta}_2 $$