This mapping is a smooth bijective mapping whose inverse is also smooth. Hence, $SO(2)$ and S^1 are diffeomorphic, and S^1 can be regarded as an embedding of $SO(2)$ in $\mathbf{R}^2$. S^1 is so easy to visualize as a unit circle that we will often use S^1 in place of $SO(2)$.

Matrices in $SO(3)$ can be mapped to vectors in $\mathbf{R}^6$ by selecting the first two columns, i.e. :

$$(r_{ij})_{i,j\in[1,3]} \in \mathbf{R}^9 \mapsto (r_{11}, r_{21}, r_{31}, r_{12}, r_{22}, r_{32}) \in \mathbf{R}^6.$$

This mapping is smooth. In addition, since the third column can be obtained as the outer product of the first two, it is bijective, and its inverse is also smooth. Thus, it allows us to embed $SO(3)$ in $\mathbf{R}^6$. However, in the rest of the book, we will not use this embedding.

4 Parameterizations

In this section we describe various ways to chose charts and to construct an atlas of $\mathcal{C}$. Once an atlas has been constructed, every configuration can be represented by a vector of m parameters.

Since the parameterization of $\mathbf{R}^N$ is trivial, we only consider $SO(N)$. We first treat the case where $N = 2$, then the case where $N = 3$. Several parameterizations other than those presented below can be found in textbooks on Kinematics (e.g. [Bottema and Roth, 1979]) and Robotics (e.g. [Paul, 1981] [Craig, 1986]).

4.1 Parameterizations of $SO(2)$

A straightforward parameterization of $SO(2)$ consists of representing an orientation of the robot $\mathcal{A}$ by a single angle θ, say the angle between the x-axes of $\mathcal{F}_{\mathcal{W}}$ and $\mathcal{F}_{\mathcal{A}}$. The map is obviously not a one-to-one map, since all the angles $\theta + 2k\pi$, with $k \in \mathbf{Z}$, represent the same orientation of $\mathcal{A}$. One way to get a chart is to restrict the values of θ to an open interval of length 2π. Two such charts, say $0 < \theta < 2\pi$ and $-\pi < \theta' < +\pi$, are needed to have an atlas. However, in most path planning algorithms, it is sufficient to consider a single interval closed at one end, say $\theta \in [0, 2\pi)$, together with modulo 2π arithmetic on θ; but this parameterization is not a chart.

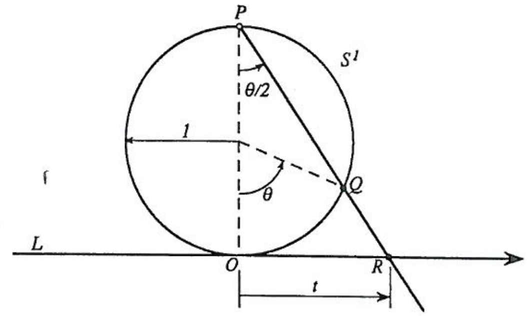

Figure 3. The stereographic projection allows us to map algebraically the circle S^1 onto the line of the reals (see text). It defines a chart valid over all S^1, except the point P. Two stereographic charts with two different points P are necessary and sufficient to build an atlas of S^1, or equivalently, $SO(2)$.

Some path planning methods require motion constraints to be expressed in algebraic form. Using the above parameterization yields expressions depending on θ through cosine and sine functions, i.e. non-algebraic functions. Another parameterization of $SO(2)$, which allows to express constraints in algebraic form, is known as the **stereographic projection**. It consists of projecting the whole unit circle S^1 (with the exception of one point) onto a line L tangent to the circle. L represents the set $\mathbf{R}$, with the origin at the point of tangency O (see Figure 3). Another line is drawn from the point P on the circle diametrically opposite the point of tangency. It intersects the circle at Q and L at R. The stereographic projection maps every point Q on the circle, except P, to the point R on L. In every open subset of S^1 not including P, the map is diffeomorphic, and hence defines a chart. Two stereographic charts, built with two different points P, are needed to form an atlas of S^1.

Let t be the abscissa of R along L and O correspond to $\theta = 0 \pmod{2\pi}$. We have:

$$t = 2\tan\frac{\theta}{2}.$$

Writing $w = \frac{t}{2}$, we can express $\cos\theta$ and $\sin\theta$ as the following rational

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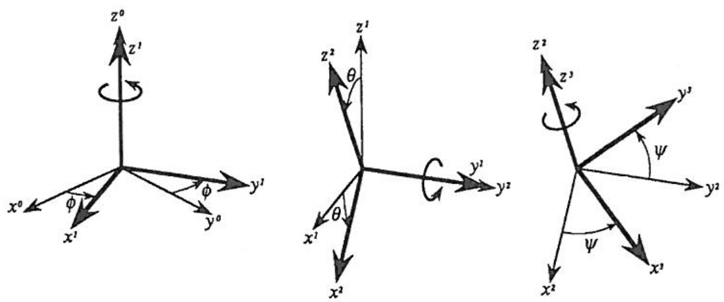

Figure 4. $\mathcal{F}^i_{\mathcal{A}} = (O_{\mathcal{A}}, x^i, y^i, z^i)$, $i = 0, 1, 2, 3$, denotes the frame $\mathcal{F}_{\mathcal{A}}$ at four successive orientations of $\mathcal{A}$. (ϕ, θ, ψ) are the Euler angles defining the orientation of $\mathcal{F}^3_{\mathcal{A}}$ relative to $\mathcal{F}^0_{\mathcal{A}}$. First, $\mathcal{A}$ is rotated by ϕ around the z^0-axis. Second, $\mathcal{A}$ is rotated by θ around the y^1-axis. Third, $\mathcal{A}$ is rotated by ψ around the z^2-axis.

fractions:

$$\cos\theta = \frac{1-w^2}{1+w^2} \quad \text{and} \quad \sin\theta = \frac{2w}{1+w^2}$$

leading to algebraic expressions in t of the constraints applying to $\mathcal{A}$'s motion.

There is no way to build an atlas of $SO(2)$ containing a single chart.

4.2 Euler Angles

The **Euler angles** are a widely used parameterization of $SO(3)$. They consist of three angles (ϕ, θ, ψ). The orientation of $\mathcal{A}$ they specify is obtained by applying three successive rotations to $\mathcal{A}$, starting at its reference orientation. Let $\mathcal{F}^0_{\mathcal{A}} = (O_{\mathcal{A}}, x^0, y^0, z^0)$ denote the frame $\mathcal{F}_{\mathcal{A}}$ at its reference orientation (see Figure 4). The rotations defined by ϕ, θ, and ψ are the following:[4]

- First, $\mathcal{A}$ is rotated by ϕ around the z^0-axis. Let $\mathcal{F}^1_{\mathcal{A}} = (O_{\mathcal{A}}, x^1, y^1, z^1)$ denote the frame $\mathcal{F}_{\mathcal{A}}$ at its new orientation.

[4] Slightly different specifications of these rotations are proposed in some textbooks. They simply determine different charts on $SO(3)$.

- Second, $\mathcal{A}$ is rotated by θ around the y^1-axis. Let $\mathcal{F}^2_{\mathcal{A}} = (O_{\mathcal{A}}, x^2, y^2, z^2)$ denote the frame $\mathcal{F}_{\mathcal{A}}$ at its new orientation.

- Third, $\mathcal{A}$ is rotated by ψ around the z^2-axis. Let $\mathcal{F}^3_{\mathcal{A}} = (O_{\mathcal{A}}, x^3, y^3, z^3)$ denote the frame $\mathcal{F}_{\mathcal{A}}$ at its new orientation.

The orientation of $\mathcal{F}^3_{\mathcal{A}}$ is the orientation of $\mathcal{A}$ defined by the Euler angles (ϕ, θ, ψ).

The parameterization of $SO(3)$ using (ϕ, θ, ψ) is not a one-to-one mapping. The same orientation is represented by $(\phi + 2k_1\pi, \theta + 2k_2\pi, \psi + 2k_3\pi)$, for any $k_1, k_2, k_3 \in \mathbf{Z}$. As for $SO(2)$, we may be tempted to apply modulo 2π arithmetic to every Euler angle. However, it is not so simple. Indeed, (ϕ, θ, ψ) represents the same orientation as $(\phi, \theta + \pi, 2\pi - \psi)$, so that a unique representation requires us to restrict the values of the three Euler angles to, say, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi)$, and $\psi \in [0, 2\pi)$. In addition, even with this restriction, there still exists a singularity at $\theta = 0$. Indeed, if $\theta = 0$, the first and third rotations are made around the same axis. Thus, any two pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2), such that $\phi_1 + \psi_1 = \phi_2 + \psi_2$, determine the same orientation of $\mathcal{A}$. This means that in the $\phi\theta\psi$-space, a motion along a line $\phi = -\psi + c \pmod{2\pi}$ at $\theta = 0$, where c is a constant, corresponds to no motion of $\mathcal{A}$.

A single set of Euler angles (ϕ, θ, ψ), with appropriate restrictions on the values of ϕ, θ and ψ, is called an Eulerian chart. Several Eulerian charts can be defined by applying three successive rotations to $\mathcal{A}$ around different axes. However, no atlas on $\mathcal{C}$ can be constructed with Eulerian charts only, since the null rotation corresponds to a singularity in all these charts. When Euler angles are used to represent the robot's orientation in a motion planner, it is usually appropriate to use a single set of Euler angles $0 \le \phi < 2\pi$, $0 \le \theta < \pi$, $0 \le \psi < 2\pi$, taking precautions at the interval extremities. If the singularities cannot be tolerated, it is preferable to use the quaternion parameterization presented in the next subsection.

Another drawback of Euler angles is that they do not allow us to express motion constraints in algebraic form. Applying a rotation specified by (ϕ, θ, ψ) to a geometric feature, say a point x, is equivalent to multiplying the coordinate vector of x by the corresponding matrix of $SO(3)$. The new coordinates of x are non-algebraic functions of ϕ, θ, and ψ. Hence, some motion planning methods which require the motion constraints to be algebraic cannot make use of this parameterization.

The matrix $\mathbf{E}_{\phi\theta\psi}$ of $SO(3)$ corresponding to the three Euler angles defined above, with the conventions taken in Section 2, is:

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi & -\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi & -\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta \\ -\sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{pmatrix}.$$

By restricting ϕ and ψ to $[0, 2\pi)$ and θ to $(0, \pi)$, and given a matrix of $SO(3)$:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the three Euler angles (ϕ, θ, ψ) are determined by the following relations:

$$\begin{aligned} \phi &= atan2(r_{23}, r_{13}) \\ \theta &= atan2(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}) \\ \psi &= atan2(r_{32}, -r_{31}) \end{aligned}$$

where $atan2(y, x)$ is defined as the argument of the complex number $x + \vec{i}y$.

4.3 Quaternions

Quaternions encode orientations in a three-dimensional workspace with four parameters.[5]

A quaternion $\mathbf{X}$ is an element of $\mathbf{R}^4$ which can be split into a scalar part x_0 and a 3-vector $\vec{x}$. To simplify notation, the scalar part and the vector part are combined using the addition symbol, i.e. $\mathbf{X} = x_0 + \vec{x}$. It

[5] The theory of quaternions is described in many textbooks. The most complete reference is [Hamilton, 1969].

is convenient to treat 3-vectors as quaternions having a zero scalar part, and scalars as quaternions having a zero vector part.

The product of two quaternions $\mathbf{P}$ and $\mathbf{Q}$ is the quaternion $\mathbf{R} = \mathbf{PQ}$ specified by:

$$\mathbf{R} = r_0 + \vec{r}$$

with:

$$r_0 = p_0 q_0 - \vec{p} \cdot \vec{q}$$

$$\vec{r} = p_0\vec{q} + q_0\vec{p} + \vec{p} \wedge \vec{q}$$

where $\vec{p} \cdot \vec{q}$ and $\vec{p} \wedge \vec{q}$ denote the inner and outer products of $\vec{p}$ and $\vec{q}$. The product $\mathbf{PQ}$ is linear in $\mathbf{P}$ and $\mathbf{Q}$.

The conjugate $\mathbf{X}^*$ of a quaternion $\mathbf{X} = x_0 + \vec{x}$ is defined by:

$$\mathbf{X}^* = x_0 - \vec{x}.$$

Notice that:

$$\mathbf{XX}^* = x_0^2 + \|\vec{x}\|^2$$

is a positive or null scalar. $\|\mathbf{X}\| = \sqrt{\mathbf{XX}^*}$ is the Euclidean length of $\mathbf{X}$ in $\mathbf{R}^4$. $\mathbf{X}$ is called a **unit quaternion** if it satisfies $\mathbf{XX}^* = 1$.

A rotation around a unit vector $\vec{n}$ by an angle θ can be represented by the following unit quaternion:

$$\mathbf{X}(\vec{n}, \theta) = \cos\frac{\theta}{2} + \vec{n}\sin\frac{\theta}{2}.$$

With this representation, one can verify that:

$$\forall a \in \mathcal{A} : a(\mathbf{q}) = \mathbf{X}_\Theta a(0) \mathbf{X}_\Theta^* + \mathcal{T}$$

where $a(\mathbf{q})$ denotes the coordinate vector of a in $\mathcal{F}_\mathcal{W}$ when $\mathcal{A}$ is at configuration $\mathbf{q} = (\mathcal{T}, \Theta) \in \mathbf{R}^3 \times SO(3)$, and $\mathbf{X}_\Theta$ denotes the unit quaternion representing the orientation Θ in $\mathbf{R}^4$. The expression $\mathbf{X}_\Theta a(0)\mathbf{X}_\Theta^* + \mathcal{T}$ is worth some explanation. In the product $\mathbf{X}_\Theta a(0)\mathbf{X}_\Theta^*$, $a(0)$ is treated as a quaternion having a zero scalar part, i.e. $0 + a(0)$. It is straightforward to verify that the product is also a quaternion having a zero scalar part. The three components of the vector part of $\mathbf{X}_\Theta a(0)\mathbf{X}_\Theta^*$ are added to the components of $\mathcal{T}$, yielding the coordinate vector $a(\mathbf{q})$.