Appendix A

Basic Mathematics

This appendix defines some mathematical terms that are used in the book. The terms are given in alphabetic order. They correspond to well-known definitions and results, and are included for the reader's convenience only.

Affine Space: Let E be a vector space. The **affine space** associated with E is a set A such that:

- Every pair $(a, b) \in A \times A$ determines a unique vector $\vec{ab} \in E$.

- Every pair $(a, \vec{x}) \in A \times E$ determines a unique element $b \in A$ such that $\vec{ab} = \vec{x}$.

- $\forall a, b, c \in A : \vec{ac} = \vec{ab} + \vec{bc}$.

The elements of A are called **points**. In general, one of them — call it O — is arbitrarily taken as the **origin**. This point determines the bijection $a \in A \mapsto \vec{x} = \vec{Oa} \in E$. It allows us to define the operations $+$ and $-$ in A:

- For any $a, b \in A$, $c = a + b$ is the element of A corresponding to $\vec{Oc} = \vec{Oa} + \vec{Ob}$. The operation is commutative.

- For any $a \in A$, $c = -a$ is the element of A corresponding to $\vec{Oc} = -\vec{Oa}$.

Bijective Map (or Bijection): Let E and F be two sets. A map $f : E \to F$ is **bijective** if and only if it is both surjective and injective.

Boundary of a Set: Let U be any subset of a topological space E. A point $x \in E$ is called a **boundary point** of U if and only if neither U, nor $E \backslash U$ is a neighborhood of x.

The set of boundary points of U is called the **boundary** of U. It is denoted by ∂U.

Class C^k: Let $f : E \subseteq \mathbf{R} \to F \subseteq \mathbf{R}$ be a function.

f is said to be of **class** C^k if all its derivatives of order less than or equal to k exist and are continuous functions on E.

f is said to be **piecewise of class** C^k if all its derivatives of order less than or equal to k exist and are continuous functions on E, except possibly at a finite number of points in any bounded subset of E.

Closure of a Set: Let U be a subset of a topological space E. The set of points of E which are not exterior points of U is called the **closure** of U. It is denoted by $cl(U)$.

Compact Set: Let E be a topological space and A a subset of E. A is **compact** if every open cover of A includes a finite subcover. In other words, if $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ is an arbitrary open cover of A, hence all U_α are open subsets of E and $\bigcup_{\alpha \in J} U_\alpha = E$, then there is a finite number of elements of J, say $\alpha_1, ..., \alpha_r$, such that $U_{\alpha_1} \cup ... \cup U_{\alpha_r} = E$.

A subset of $\mathbf{R}^n$ is compact if and only if it is both closed and bounded.

Connectedness: A topological space is **connected** if it is not the union of two non-empty, open, disjoint subspaces.

Continuous Map: Let E and F be two topological spaces. A map

$f : E \to F$ is **continuous** if and only if the inverse image of any open set (in the topology of F) is an open set (in the topology of E).

Convex Set: A set $S \subset \mathbf{R}^n$ is **convex** if for any two points $x_1, x_2 \in S$ the segment $\{x \ / \ x = \lambda x_1 + (1-\lambda)x_2 \ ; \ 0 \leq \lambda \leq 1\}$ is entirely contained in S.

Diffeomorphism: Let $f : X \subset \mathbf{R}^n \to Y \subset \mathbf{R}^m$ be a map between two subsets of two Euclidean spaces. It is a **diffeomorphism** if f is bijective and smooth, and the inverse map $f^{-1} : Y \to X$ is also smooth. X and Y are said to be **diffeomorphic** if such a map exists.

Distance: Let E be a set. A **distance** or **metric** on E is a function $d : E \times E \to \mathbf{R}$ verifying the following axioms:

- $\forall x, y \in E : d(x,y) \geq 0$ (positiveness).

- $d(x,y) = 0 \Rightarrow x = y$ (non-degeneracy).

- $\forall x, y \in E : d(x,y) = d(y,x)$ (symmetry).

- $\forall x, y, z \in E : d(x,y) + d(y,z) \geq d(x,z)$ (triangle inequality).

The pair (E, d) is called a **metric space**. When there is no ambiguity about the distance, E alone is called the metric space.

Let $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ be two points of $\mathbf{R}^n$. The map:

$$\|\mathbf{x} - \mathbf{y}\| = [(y_1 - x_1)^2 + (y_2 - x_2)^2 + ... + (y_n - x_n)^2]^{1/2}$$

defines a distance in $\mathbf{R}^n$ called the **Euclidean distance**. The set $\mathbf{R}^n$ equipped with this distance is a Euclidean space.

Equivalence (Relation of): A **relation of equivalence** on a set E is any subset $\mathcal{R} \subseteq E \times E$ such that:

- $\forall x \in E : (x,x) \in \mathcal{R}$ (reflexivity).

- $\forall x, y \in E : (x,y) \in \mathcal{R} \Rightarrow (y,x) \in \mathcal{R}$ (symmetry).

- $\forall x, y, z \in E : (x,y) \in \mathcal{R}$ and $(y,z) \in \mathcal{R} \Rightarrow (x,z) \in \mathcal{R}$ (transitivity).

Exterior of a Set: Let U be any subset of a topological space E. A point $x \in E$ is called an **exterior point** of U if and only if $E \backslash U$ is a neighborhood of x.

The set of the exterior points of U is called the **exterior** of U.

Group: A set G together with a binary operation $\diamond : G \times G \to G$ has the structure of a **group** if and only if:

- $\diamond$ is associative, i.e. $x \diamond (y \diamond z) = (x \diamond y) \diamond z$ for all x, y and z in G.

- There exists $e \in G$ such that $x \diamond e = x$, for all x in G. The element e is called the *identity*.

- For each element x in G there exists an element x' in G such that $x \diamond x' = x' \diamond x = e$. The elements x and x' are said to be the *inverses* of each other.

In addition, G is said to be a *commutative* or *abelian* group if and only if $x \diamond y = y \diamond x$ for all x and y in G.

Homeomorphism: Let $f : X \subset \mathbf{R}^n \to Y \subset \mathbf{R}^m$ be a map between two subsets of two topological spaces. It is a **homeomorphism** if f is bijective and continuous, and the inverse map $f^{-1} : Y \to X$ is also continuous (in the subspace topologies of X and Y). X and Y are said to be **homeomorphic** if such a map exists.

Implicit Function Theorem: Let $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$ be a smooth map in an open subset X of $\mathbf{R}^n \times \mathbf{R}^m$. Suppose $(a,b) \in X$, with $a \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, and $f(a,b) = 0$. Let M be the $m \times m$ matrix:

$$\left(D_{n+j} f^i(a,b)\right)_{1 \leq i,j \leq m}$$

where $D_{n+j} f^i(a,b)$ denotes the first derivative at (a,b) of the i^{th} component of f with respect to the $(n+j)^{\text{th}}$ variable of f.

If $det\ M \neq 0$, there is an open set $A \subset \mathbf{R}^n$ containing a and an open set $B \subset \mathbf{R}^m$ containing b, with the following property: For each $x \in A$ there is a unique $g(x) \in B$ such that $f(x, g(x)) = 0$. The function g is smooth.

Injective Map: Let E and F be two sets. A map $f : E \to F$ is **injective** if and only if:

$$\forall x, x' \in E : f(x) = f(x') \Rightarrow x = x'$$

or, equivalently:

$$\forall x, x' \in E : x \neq x' \Rightarrow f(x) \neq f(x').$$

Inner Product: Let E be a vector space over $\mathbf{R}$. An **inner product** $<,>$ on E is a map of $E \times E$ into $\mathbf{R}$:

$$(\vec{x}, \vec{y}) \in E \times E \mapsto <\vec{x}, \vec{y}> \in \mathbf{R}$$

such that:

- $\forall \vec{x} \in E : <\vec{x}, \vec{x}> \geq 0$.
- $\forall \vec{x} \in E : <\vec{x}, \vec{x}> = 0 \Leftrightarrow \vec{x} = \vec{0}$.
- $\forall \vec{x}, \vec{y} \in E : <\vec{x}, \vec{y}> = <\vec{y}, \vec{x}>$.
- $\forall \vec{x_1}, \vec{x_2}, \vec{y} \in E, \forall \lambda_1, \lambda_2 \in \mathbf{R}$:

$$<\lambda_1 \vec{x_1} + \lambda_2 \vec{x_2}, \vec{y}> = \lambda_1 <\vec{x_1}, \vec{y}> + \lambda_2 <\vec{x_2}, \vec{y}>.$$

Interior of a Set: Let U be any subset of a topological space E. A point $x \in E$ is called an **interior point** of U if and only if U is a neighborhood of x.

The set of the interior points of U is called the **interior** of U. It is denoted by $int(U)$.

Isomorphism: Let E be a set equipped with a binary operation $\diamond : E \times E \to E$. Let F be a set equipped with a binary operation $\triangle : F \times F \to F$. Let $f : E \to F$ be a bijective map between the two sets.

The map f is said to be an **isomorphism** if and only if:

$$\forall x, y \in E : f(x \diamond y) = f(x) \triangle f(y).$$

Since f is bijective, if it is an isomorphism, we also have:

$$\forall x, y \in F : f^{-1}(x \triangle y) = f^{-1}(x) \diamond f^{-1}(y).$$

If there exists an isomorphism f between E and F, E and F are said to be **isomorphic**.

Manifold: A topological space M is a **manifold** (resp. a **smooth manifold**) if every point $x \in M$ has an open neighborhood homeomorphic (resp. diffeomorphic) to an open ball of $\mathbf{R}^m$, for some m independent of x. The number m is the **dimension** of the manifold.

Manifold with Boundary: A subset M of a topological space X is an m-dimensional **manifold with boundary** if every point $x \in M$ has a neighborhood V (in the topology of X) such that the set $V \cap M$ is homeomorphic to either an open ball of $\mathbf{R}^m$ or a closed half-space of $\mathbf{R}^m$.

Measure Zero (Set of): A subset E of $\mathbf{R}^n$ has **measure zero** if it can be covered by a countable number of rectangloids (Cartesian products of n intervals in $\mathbf{R}$) with arbitrarily small total volume.

Metric: See **Distance**.

Metric Topology: Let (E, d) be a metric space. The topology on E induced by the distance d, called the **metric topology**, is the set $\mathcal{O}$ of all subsets $V \subset E$ such that:

$$\forall x \in V, \exists \varepsilon > 0 : B_\varepsilon(x) \subset V$$

where $B_\varepsilon(x) = \{y \in E \,/\, d(x, y) < \varepsilon\}$ is the open ball of radius ε centered at x.

Two distances d_1 and d_2 in E *induce the same topology* if and only if for any pair x and x' of elements of E:

- $\forall \varepsilon > 0, \exists \eta > 0$ such that: $d_1(x, x') < \eta \Rightarrow d_2(x, x') < \varepsilon$, and
- $\forall \eta > 0, \exists \varepsilon > 0$ such that: $d_2(x, x') < \varepsilon \Rightarrow d_1(x, x') < \eta$.

Minkowski Operators: Let X be an affine space whose origin is O, and A and B be any two subsets of X. The Minkowski operators $\oplus$

(addition) and $\ominus$ (subtraction) are defined by:

$$\begin{aligned} A \oplus B &= \{x \,/\, x = a + b, a \in A, b \in B\}, \\ \ominus B &= \{-b \,/\, b \in B\}, \\ A \ominus B &= A \oplus (\ominus B). \end{aligned}$$

Neighborhood: Let E be a topological space and x an element of E. A subset $U \subset E$ is called a **neighborhood** of x if and only if there is an open set V (in the topology of E) such that $x \in V \subset U$.

Norm: Let E be a vector space over $\mathbf{R}$. A **norm** on E is a map:

$$\vec{x} \in E \mapsto \|\vec{x}\| \in \mathbf{R}$$

such that:

- $\forall \vec{x} \in E : \|\vec{x}\| \geq 0$.
- $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$.
- $\forall \vec{x} \in E, \forall \lambda \in \mathbf{R} : \|\lambda \vec{x}\| = |\lambda| \, \|\vec{x}\|$.
- $\forall \vec{x}, \vec{y} \in E : \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

If an inner product $<\,,\,>$ has been specified on E, then the map:

$$\vec{x} \in E \mapsto \sqrt{<\vec{x}, \vec{x}>} \in \mathbf{R}$$

defines a norm on E. By definition, a **Euclidean vector space** is a vector space in which one has defined an inner product and the norm is the norm associated with this inner product.

Outer Product: Let $\vec{x}$ and $\vec{y}$ be two vectors of the Euclidean space $\mathbf{R}^2$ whose components in an orthonormal basis are x_1 and x_2, and y_1 and y_2, respectively. The **outer product** of $\vec{x}$ and $\vec{y}$, denoted by $\vec{x} \wedge \vec{y}$, is the scalar z defined by:

$$z = x_1 y_2 - x_2 y_1.$$

Let $\vec{x}$ and $\vec{y}$ be two vectors of the Euclidean space $\mathbf{R}^3$, whose components in an orthonormal basis β are x_1, x_2 and x_3, and y_1, y_2 and y_3, respectively. The **outer product** of $\vec{x}$ and $\vec{y}$, denoted by $\vec{x} \wedge \vec{y}$, is the vector $\vec{z} \in \mathbf{R}^3$, whose components z_1, z_2 and z_3 in β are defined by:

$$\begin{aligned} z_1 &= x_2 y_3 - x_3 y_2, \\ z_2 &= x_3 y_1 - x_1 y_3, \\ z_3 &= x_1 y_2 - x_2 y_1. \end{aligned}$$

Path-Connectedness: A topological space E is **path-connected** if any two elements of E are connected by a path in E.

In a manifold, connectedness and path-connectedness are two equivalent notions.

Quotient Topology: Let X be a topological space and $\sim$ an equivalence relation on X. The quotient space $X/\sim$ can be given the structure of a topological space as follows: A subset U of $X/\sim$ is open if and only if the subset of X containing the elements of the equivalent classes in U is an open subset of X. This topology is called the **quotient topology** induced by the topology of X and the relation $\sim$.

Regular Set: Let Y be a subset of a topological space X. Y is said to be a **regular set** if and only if it is equal to the closure of its interior, i.e. $Y = cl(int(Y))$.

Retraction: Let X be a topological space and A be a subset of it. A **retraction** from X onto A is a surjective continuous map $X \to A$ whose restriction to A is the identity.

Subspace Topology: Let $(E, \mathcal{O})$ be a topological space and F a subset of E. The set $\mathcal{O}' = \{X \cap F \,/\, X \in \mathcal{O}\}$ is a topology on F and is called the **subspace topology**. The topological space $(F, \mathcal{O}')$ is called a **subspace** of $(E, \mathcal{O})$.

Surjective Map: Let E and F be two sets. A map $f : E \to F$ is **surjective** if and only if:

$$\forall y \in F, \exists x \in E : y = f(x).$$

f is also said to be a map of E **onto** F.

Topology: A **topology** on a set E is a set $\mathcal{O}$ of subsets of E such that:

- Any union of subsets in $\mathcal{O}$ belongs to $\mathcal{O}$.
- The intersection of any two subsets in $\mathcal{O}$ belongs to $\mathcal{O}$.
- $\emptyset$ and E belong to $\mathcal{O}$.

The pair $(E, \mathcal{O})$ is called a **topological space**. When there is no ambiguity about the topology, E alone is called the topological space.

The elements of $\mathcal{O}$ are called **open sets**. Every set $E \backslash U$, where $U \in \mathcal{O}$, is called a **closed set**.

Vector Space: A set E is a **vector space** over $\mathbf{R}$ if it has:

- A binary operation $E \times E \rightarrow E$, called $+$, with which it is an abelian group, i.e. :
  - $\forall x, y \in E : x + y = y + x$,
  - $\forall x, y, z \in E : x + (y + z) = (x + y) + z$,
  - $\exists e \in E, \forall x \in E : x + e = x$,
  - $\forall x \in E, \exists x' \in E : x + x' = e$.
- An operation $E \times \mathbf{R} \rightarrow E$, called $\times$, such that:
  - $\forall x, y \in E, \forall \lambda \in \mathbf{R} : \lambda \times (x + y) = \lambda \times x + \lambda \times y$,
  - $\forall x \in E, \forall \lambda, \mu \in \mathbf{R} : (\lambda + \mu) \times x = \lambda \times x + \mu \times x$,
  - $\forall x \in E, \forall \lambda, \mu \in \mathbf{R} : (\lambda\mu) \times x = \lambda \times (\mu \times x)$,
  - $\forall x \in E : 1 \times x = x$.

Often, $\lambda \times x$ is written λx, the elements of E are denoted by $\vec{x}$, and the identity element e in E is written $\vec{0}$.

Weierstrass' Approximation Theorem: Any continuous function $f : \Delta \subset \mathbf{R} \rightarrow \mathbf{R}$, where Δ is a compact interval of $\mathbf{R}$, is the limit of a sequence of polynomials that uniformly converges in Δ.

Appendix B

Computational Complexity

This appendix briefly reviews important concepts in Computational Complexity. A more comprehensive presentation of these concepts can be found in various books such as [Garey and Johnson, 1979] [Aho, Hopcroft and Ullman, 1983] and [Sedgewick, 1988].

The most important (and most commonly used) measure of the performance of an algorithm A is its **running time**, i.e. the time required by its execution. This time should be evaluated as a function of the input of A.

In general, we want to estimate the running time, not as a function of the exact input, which would be too specific, but as a function of one or a few parameters measuring the **size** of the input. In motion planning, typical parameters for measuring the size of a problem are the dimension of the configuration space, the number of polynomial constraints defining the shape of the objects, and the maximal degree of the polynomials used to express these constraints.

Let us assume that the size of the input to A is measured by a single parameter denoted by n. Let $T(n)$ denote the running time of A as a