$O(n)$	order of n
J	Jacobian
Γ	Christoffel symbol
RM	roadmap
$\mathcal{W}$	workspace
$\mathcal{Q}$	configuration space
$\mathcal{Q}_{\text{free}}$	free space
$x(k)$	state at time k
$\|\|x\|\|$	norm of x
$\subseteq$	subset of
$\subset$	strict subset of
$\text{cl}(A)$	closure of A
T^n	n-dimensional torus
S^n	n-dimensional sphere in $\mathbb{R}^{n+1}$
$SO(n)$	special orthogonal group
$SE(n)$	special Euclidean group
$B_\epsilon(q)$	open ball of radius ϵ centered at q
Df	differential of f
∇f	gradient of f
∇	affine connection
$\nabla_{Y_1} Y_2$	covariant derivative of Y_2 with respect to Y_1
C^0	continuous
C^n	n times differentiable
$\langle x, y \rangle$	inner product of x and y
$\mathcal{I}$	identity matrix
$\text{atan2}(y, x)$	returns angle to (x, y) in the plane in range $[-\pi, \pi)$
$T_x\mathcal{M}$	tangent space of $\mathcal{M}$ at x
$T\mathcal{M}$	tangent bundle of $\mathcal{M}$
$[f, g]$	Lie bracket of vector fields f, g
$\overline{\text{Lie}}(\mathcal{G})$	the Lie algebra of a set of vector fields $\mathcal{G}$
$\overline{\mathcal{D}}$	involutive closure of the distribution $\mathcal{D}$
$\mathcal{U}_\pm$	control set positively spanning $\mathbb{R}^m$
$\mathcal{U}_+$	control set spanning $\mathbb{R}^m$
$\langle Y_1 : Y_2 \rangle$	the symmetric product of vector fields Y_1 and Y_2
$\overline{\text{Sym}}(\mathcal{Y})$	the symmetric closure of the distribution $\mathcal{Y}$
$\text{span}(\{x_1, \ldots, x_n\})$	the linear span of $\{x_1, \ldots, x_n\}$

CHOSET

B Basic Set Definitions

CONSIDER A collection of elements called a *set*. The plane is a set; the real line is a set; a point is a set; the unit interval is a set. Sets can also be listed as collections of elements, e.g., $S_1 = \{1, 4, 9\}$ and $S_2 = \{\text{cow, chicken, pig}\}$ are both sets. The collection of these sets is also a set, i.e., $\{\mathbb{R}^2, \mathbb{R}, [0, 1]\}$ is a set. Given two sets A and B, A is said to be a *subset* of B (denoted $A \subset B$) if every element of A is also an element of B. Of the two examples above, S_1 is a subset of the set of positive integers and S_2 is a subset of the set of animals.

Given $A \subset B$, the *complement* of A in B (denoted $B \backslash A$) is defined to be all of the elements of B that are not in A, i.e.,

$$B \backslash A = \{x \mid x \in B, x \notin A\}.$$

The *union* of A and B (denoted $A \cup B$) is to be the set of points that is in either A or B, i.e.,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B (denoted $A \cap B$) is defined to be the set of all points that are in both A and B, i.e.,

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

For the remainder of this appendix we restrict the discussion to sets that are subsets of $\mathbb{R}^n$ for some n. Consider a point $x \in \mathbb{R}^n$, and define an ϵ-*neighborhood* of x to be the set

$$B_\epsilon(x) = \{y \in \mathbb{R}^n \mid d(x, y) < \epsilon\}.$$

The set $B_\epsilon(x)$ is also sometimes called an *open ball* of radius ϵ around the point x. We also sometime use the word *neighborhood* to refer to an ϵ-neighborhood with ϵ arbitrarily small.

A set $A \subset \mathbb{R}^n$ is said to be *open* if, for every point x in A, there is some ϵ so that $B_\epsilon(x)$ is also contained in A. A set A is said to be *closed* if its complement is open. Note that the concept of closure depends on the the ambient space. The set $\mathbb{R}^m$ considered by itself is open. But if $m < n$ and we consider $\mathbb{R}^m$ as a subset of the ambient space $\mathbb{R}^n$, then $\mathbb{R}^m$ is closed since its complement $\mathbb{R}^n \backslash \mathbb{R}^m$ is open. By the same token, when considered as a subset of the plane, the interval $(0, 1)$ is neither closed nor open.

The following definitions derive from open and closed sets for ACS:

DEFINITION B.0.2 (Closure/Interior/Boundary)

- Closure *of* A, *denoted* $\text{cl}(A)$, *is the intersection of all closed sets containing* A.
- Interior *of* A, *denoted* $\text{int}(A)$, *is the union of all open sets contained in* A.
- Boundary *of* A, *denoted* ∂A, *is* $\text{cl}(A) \bigcap \text{cl}(S \backslash A)$.

EXAMPLE B.0.3 *Consider* $[0, 1]$ *as a subset of* $\mathbb{R}^1$.

$$\text{cl}([0,1]) = [0,1]$$
$$\text{int}([0,1]) = (0,1)$$
$$\begin{aligned}\partial[0,1] &= [0,1] \bigcap \text{cl}\left((-\infty, 0) \bigcup (1, \infty)\right) \\ &= [0,1] \bigcap ((-\infty, 0] \bigcup [1, \infty)) \\ &= \{0, 1\}\end{aligned}$$

The following demonstrate how union and intersection operate on closures and interiors:

$$\begin{aligned}
A \subset B \quad &\Rightarrow \text{int}(A) \subset \text{int}(B) \text{ and } \text{cl}(A) \subset \text{cl}(B) \\
A \subset S \quad &\Rightarrow S \backslash \text{cl}(A) = \text{int}(S \backslash A),\ S \backslash \text{int}(A) = \text{cl}(S \backslash A). \\
A \subset S \quad &\Rightarrow \text{cl}(\emptyset) = \text{int}(\emptyset) = \emptyset,\ \text{cl}(S) = \text{int}(S) = S \\
&\quad \text{cl}(\text{cl}(A)) = \text{cl}(A) \\
&\quad \text{cl}(A \bigcup B) = \text{cl}(A) \bigcup \text{cl}(B),\ \text{int}(A) \bigcup \text{int}(B) \subset \text{int}(A \bigcup B) \\
&\quad \text{cl}(A \bigcap B) \subset \text{cl}(A) \bigcap \text{cl}(B),\ \text{int}(A \bigcap B) = \text{int}(A) \bigcap \text{int}(B).
\end{aligned}$$

A subset A of B is *dense* if $\text{cl}(A) = B$. So $(0, 1)$ is dense in $[0, 1]$ because $\text{cl}(0, 1) = [0, 1]$. Intuitively, a subset A of B is dense if A is "almost as big" as B. The open interval and closed interval both have length 1. The set $[0, 1] \backslash \{.5\}$ is dense in $[0, 1]$. Intuitively, this means that taking away one point from an interval does not affect the size of the interval. The set of rational numbers, i.e., the set of real numbers that can be written as a fraction of two integers, is dense in the real line. A line is *not* dense in the plane. The plane, with a line removed from it, is dense in the plane.

We can also define a notion of subtraction and addition of sets. The *Minkowski sum* of A and B is

$$A \oplus B = \{x \mid x = a + b, a \in A, b \in B\}.$$

The *Minkowski difference* is

$$A \ominus B = \{x \mid x = a - b, a \in A, b \in B\}.$$

Two points in a set A are said to be within *line of sight* of each other if the straight line segment connecting them is completely contained in A. Line of sight is related to convexity. A set A is *convex* if for every $x, y \in A$, the line segment

$$\{tx + (1 - t)y \mid t \in [0, 1]\}$$

is contained in A. The *convex hull* of a set A is denoted as $\text{Co}(A)$ and is defined to be the smallest convex set that contains A. If $A \subset \mathbb{R}^n$ is a finite set with m elements $\{x_1, x_2, \ldots, x_m\}$, we can express

$$\text{Co}(A) = \left\{ y = \sum_{i=1}^{m} a_i x_i \,\middle|\, a_i \geq 0 \text{ for all } i;\ \sum_{i=1}^{m} a_i = 1 \right\}.$$

A set A is said to be *star-shaped* if there exists an $x \in A$ such that for every $y \in A$ the line segment $\{tx + (1 - t)y \mid t \in [0, 1]\}$ is contained in A. In other words, all points in A are within line of sight of at least one common point. All convex sets are star-shaped, but the converse is not true.

C Topology and Metric Spaces

C.1 Topology

OPERATORS act on elements of sets. In appendix B, the set complement operator was defined with respect to a superset S. Furthermore, the definitions of open and closed sets were predicated on one definition: the open neighborhood. Now we are going to reverse things. An open neighborhood will be defined in terms of open sets, and a topological space will be defined in terms of its set of elements and its open sets. This appendix is meant to be introductory. See, e.g., [9] for a complete discussion of these topics.

DEFINITION C.1.1 (Topology) *A* topological space *is a set S together with a collection O of subsets called* open sets *such that*

- *$\emptyset \in O$ and $S \in O$,*
- *if $U_1, U_2 \in O$, then $U_1 \bigcap U_2 \in O$,*
- *the union of any collection of open sets is an open set.*

Open sets can be arbitrarily designed as long as they satisfy the above three properties. The *standard topology* on $\mathbb{R}^m$ has $S = \mathbb{R}^m$ with O containing $\mathbb{R}^m$, the empty set $\emptyset$, all open rectangles, and their unions. An example is the real line with open intervals, i.e., $S = \mathbb{R}$, with O consisting of any open interval, the union of open

intervals, $\mathbb{R}$, and $\emptyset$. To show this we look to the three conditions in definition C.1.1:

- $\mathbb{R}, \emptyset \in O$ by definition,
- $(a, b) \in O$ and $(c, d) \in O$, so

$$(a, b) \bigcap (c, d) = \begin{cases} (c, b) \in O & \text{or,} \\ (a, d) \in O & \text{or,} \\ \emptyset \in O, & \end{cases}$$

- any finite or infinite union of open intervals is an open interval.

The *trivial topology* on a set S consists of $O = \{\emptyset, S\}$. The *discrete topology* of a set S is defined by $O = \{A \mid A \subset S\}$. That is, the open sets are everything.

Now the definition of the open neighborhood stems from the definition of open sets. The definitions of closed sets, closure, boundary, interior, and denseness remain the same.

DEFINITION C.1.2 *A* neighborhood *of a point x, denoted* nbhd(x), *is an open set that contains x.*

C.2 Metric Spaces

The open sets of a topological space can be constructed using a distance function. In $\mathbb{R}^m$, the standard Euclidean distance function

$$d(x, y) = \left(\sum_{i=1}^{m} (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

defines open sets that are open balls. More generally,

DEFINITION C.2.1 (Metric Space) *Let M be a set. A* metric *on M is a function $d : M \times M \to \mathbb{R}^{\geq 0}$ such that for all $m_1, m_2, m_3 \in M$,*

1. *(Definiteness) $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$*
2. *(Symmetry) $d(m_1, m_2) = d(m_2, m_1)$, and*
3. *(Triangle inequality) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$.*

A metric space *is the pair (M, d).*

Note that the intuitive notion that distance must be non-negative follows directly from the three conditions above. Specifically, condition 3 allows us to write $d(m_1, m_1) \leq d(m_1, m_2) + d(m_2, m_1)$. The left-hand side of this expression is zero by condition 1 and the right-hand side is $2d(m_1, m_2)$ by condition 2, yielding $d(m_1, m_2) \geq 0$.

For $\epsilon > 0$ and $m \in M$, the *open ball* centered at m is defined to be

$$B_\epsilon(m) = \{n \in M \mid d(m, n) < \epsilon\}.$$

The set of all open balls and the union of open balls forms the *metric topology* on the metric space (M, d).

There are many distance functions other than the standard Euclidean metric. For example, the *Manhattan distance metric* is defined to be

$$d(x, y) = \sum_{i=1}^{m} |x_i - y_i|.$$

This metric is so named because it measures how far a taxicab must drive in a city grid to get from one location to another. Different metrics can be used to induce the same topology. The Manhattan and standard Euclidean metrics induce the same topology. Two metrics induce the same topology if, for any open ball at x by the first metric, there is an open ball by the second metric contained completely in the first ball, and vice versa.

C.3 Normed and Inner Product Spaces

A metric space is a special case of a topological space. Next we introduce a normed space, which is a special case of a metric space. We also introduce an inner product space, which is a special case of a normed space.

DEFINITION C.3.1 *A normed space E is a subset of a metric space M that has an operator $\|\cdot\| : E \to \mathbb{R}$ such that*

- $\|e\| \geq 0$ *for all $e \in E$, and $\|e\| = 0$ if and only if e is the zero vector (positive definiteness),*
- $\|\lambda e\| = |\lambda| \|e\|$ *for all $e \in E$ and $\lambda \in \mathbb{R}$ (homogeneity),*
- $\|e_1 + e_2\| \leq \|e_1\| + \|e_2\|$ *for all $e_1, e_2 \in E$ (triangle inequality).*

The norm can be used to define the open sets and induce a metric. A sequence $\{x_1, x_2, x_3, \ldots\}$ is said to be a *Cauchy sequence* if for any $\epsilon > 0$ there exists an

integer k such that $\|x_i - x_j\| < \epsilon$ for all $i, j > k$. When a normed space has a corresponding metric for which every Cauchy sequence converges to a point in the space, we term this space a *Banach space*.

DEFINITION C.3.2 *An* inner product *on a real vector space E is a mapping $\langle \cdot, \cdot \rangle : E \times E \to \mathbb{R}$ such that*

- $\langle e, e_1 + e_2 \rangle = \langle e, e_1 \rangle + \langle e, e_2 \rangle$,
- $\langle e, \alpha e_1 \rangle = \alpha \langle e, e_1 \rangle$,
- $\langle e_2, \alpha e_1 \rangle = \langle e_1, \alpha e_2 \rangle$,
- $\langle e, e \rangle \geq 0$ *and* $\langle e, e \rangle = 0$ *if and only if e is zero.*

An inner product induces the norm $\|e\| = \langle e, e \rangle$, and a norm in turn induces a metric. When an inner product space has a corresponding metric for which every Cauchy sequence converges, we call this space a *Hilbert space*.

C.4 Continuous Functions

Paths are defined in terms of a continuous function. Let $f : S \to T$ be a mapping from the *domain* S to the *range* T. The points $f(s)$ are the *values* of f, where $s \in S$. If $U \subset S$, then the *image* of U under f is denoted $f(U) = \{f(x) \in T \mid x \in U\}$. If $V \subset T$, then the *preimage* of V under f is denoted $f^{-1}(V) = \{x \in S \mid f(x) \in V\}$. First, we introduce an abstract notion of a continuous function and then specialize it for metric spaces.

DEFINITION C.4.1 *Let S and T be topological spaces and $f : S \to T$ be a mapping. f is* continuous *at $u \in S$ if for every $V = \text{nbhd}(f(u))$ there is a $U = \text{nbhd}(u)$ such that $f(U) \subset V$. The mapping f is continuous if for every open subset $V \subset T$, $f^{-1}(V) = \{u \in S \mid f(u) \in V\}$ is open in S.*

Essentially, a continuous function is a function where the preimage of an open set is an open set. Now we introduce the standard "delta-epsilon" method for defining continuous functions on metric spaces: The function f is continuous at s if for every $\epsilon > 0$ there exists a $\delta > 0$ where

(C.1) $$d(x, s) < \delta \text{ implies } d(f(x), f(s)) < \epsilon.$$

EXAMPLE C.4.2 (Continuous Function) *Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^2$. In order to show that f is continuous at a point s, we must find a $\delta > 0$ such that $d(f(x), f(s)) < \epsilon$ for arbitrarily small $\epsilon > 0$. Note that in $\mathbb{R}$ the distance function is $d(x, s) = |x - s|$. First, we study the quantity $|f(x) - f(s)|$:*

$$\begin{aligned} |f(x) - f(s)| &= |x^2 - s^2| \\ &= |x - s||x + s| \\ &= |x - s||x - s + 2s| \end{aligned}$$

Using the triangle inequality, we get

$$|f(x) - f(s)| \leq |x - s|(|x - s| + 2|s|).$$

Now we can subsitute $|x - s| < \delta$ to see that $|f(x) - f(s)|$ will be less than ϵ if

$$\delta(\delta + 2|s|) < \epsilon.$$

Using the quadratic formula, we see that this inequality can be satisfied for

$$\delta < -|s| + \sqrt{s^2 + \epsilon}.$$

The term on the right-hand side of this inequality is positive, so we can find a suitable δ. This proves that the function $f(x) = x^2$ is continuous at any point $s \in \mathbb{R}$. Note the that choice of δ depends on both s and ϵ.

The set of continuous functions is denoted C^0. If the derivative of a continuous function f is continuous, then f is said to be differentiable and belongs to a set denoted C^1. If c is k-wise differentiable, then it belongs to a set denoted C^k. If all derivatives of f exist, then f belongs to C^∞ and f is said to be *smooth*. While a path is only required to be of class C^0, a trajectory must belong to C^k, $k > 0$, to allow the definition of velocity, acceleration, etc., at all points where the system is moving.

The following are equivalent statements:

$$f : S \to T \text{ is continuous.} \iff f(\text{cl}(A)) \subset \text{cl}(f(A)) \text{ for } A \subset S$$
$$\iff f^{-1}(\text{int}(B)) \subset \text{int}(f^{-1}(B)) \text{ for } B \subset T$$

Finally, another useful property of continuous functions is that things "change" continuously. Specifically, if the scalar functions f and g are continuous at x and $f(x) < g(x)$, then there exists a nbhd(x) such that for all $y \in$ nbhd(x), $f(y) < g(y)$.

C.5 Jacobians and Gradients

Consider a vector-valued function $f : \mathbb{R}^m \to \mathbb{R}^n$ where f can be written

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{bmatrix},$$

where $f_i : \mathbb{R}^m \to \mathbb{R}$ for $i \in \{1, 2, \ldots, n\}$.

We define the *differential* of f to be the matrix[1]

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{bmatrix}.$$

The matrix Df is denoted in a number of different ways. It is sometimes called the *Jacobian* of f and denoted J (see chapter 3). It is sometimes called the *tangent map* of f and denoted Tf. Sometimes it is necessary to specify which variables are used in the differentiation. Hence the differential can also be denoted $\frac{\partial f}{\partial x}$. Putting the variable name in the subscript serves a similar purpose. The symbols $D_x f$, J_x, and $T_x f$ all denote the differential of f with respect to the variable x.

Given a function $g : \mathbb{R}^n \to \mathbb{R}$, the *gradient* of g is defined to be

$$\nabla g = \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \vdots \\ \frac{\partial g}{\partial x_n} \end{bmatrix}.$$

As in the case of the differential, the notation $\nabla_x g$ is sometimes used to make explicit the fact that g is differentiated with respect to x. The vector $\nabla g(x)$ points in the direction that maximally increases the function at the point x. Note that by this defintition $\nabla f(x) = Df^T$. The decision as to whether the gradient should be a row vector or a column vector is somewhat arbitrary. In this book we define it as a column vector because that is the convention commonly used in the robotics community when discussing planning algorithms based on artificial potential fields.

1. To be technically accurate, the differential is actually a map from the tangent space of $\mathbb{R}^m$ (which happens to also be $\mathbb{R}^m$) to the tangent space of $\mathbb{R}^n$ (which is $\mathbb{R}^n$). For the purposes of this appendix, we simply represent Df as an $n \times m$ matrix.