is considerably more general and can be induced by other metrics (see Section 7), including metrics which are independent of the representation of $\mathcal{C}$, such as the following distance function:

$$d(\mathbf{q}, \mathbf{q}') = \max_{a \in \mathcal{A}} \|a(\mathbf{q}) - a(\mathbf{q}')\|$$

where $\|x - x'\|$ denotes the Euclidean distance between any two points $x$ and $x'$ in a Euclidean space. All these metrics share the following "natural" property: The distance between two configurations $\mathbf{q}$ and $\mathbf{q}'$ tends toward zero when the regions $\mathcal{A}(\mathbf{q})$ and $\mathcal{A}(\mathbf{q}')$ tend to coincide, and conversely. In the following, we will consider the topology of $\mathcal{C}$ defined above independently of any particular metric.

The subset $\mathcal{M}_{\mathcal{C}}$ of $\mathbf{R}^M$ is defined by $\frac{1}{2}N(N+1)$ polynomial equations $h_k(u_1, u_2, ..., u_M) = 0$, with $k = 1$ to $\frac{1}{2}N(N+1)$, where $u_1, u_2, ..., u_M$ are merely new symbols for $x, ..., r_{11}, ..., r_{NN}$. (Actually, there is an additional equation: the equation that selects the matrices with determinant +1; this equation, however, has no effect on the local properties of $\mathcal{M}_{\mathcal{C}}$.) One can verify that, whether $N = 2$ or 3, the smooth functions $h_k$ admit gradient vectors

$$\vec{\nabla} h_k = \left(\frac{\partial h_k}{\partial u_i}\right)_{i=1,...,M}$$

which are linearly independent at every $(u_1, ..., u_M) \in \mathcal{M}_{\mathcal{C}}$. In other words, $\mathcal{M}_{\mathcal{C}}$ is at the intersection of $\frac{1}{2}N(N+1)$ hypersurfaces of dimension $M - 1$, such that at every intersection point, the tangent planes of these surfaces are distinct.

Let $m = M - \frac{1}{2}N(N+1) = \frac{1}{2}N(N+1)$. Every point $\mathbf{u} = (u_1, ..., u_M)$ in $\mathcal{M}_{\mathcal{C}}$ admits an open neighborhood $U$ diffeomorphic to an open subset $X$ of $\mathbf{R}^m$. Indeed, according to the Implicit Function Theorem (see Appendix A), there exists an open neighborhood $U$ of $\mathbf{u}$ in $\mathcal{M}_{\mathcal{C}}$ with the following property: $\frac{1}{2}N(N+1) = M - m$ variables $u_i$, say $u_{m+1}, ..., u_M$, can be expressed as smooth functions $g_i$, with $i = m+1, ..., M$, of the $m$ other variables $u_1, ..., u_m$; these functions are unique. The map $\phi : U \subset \mathcal{M}_{\mathcal{C}} \rightarrow \mathbf{R}^m$ defined by:

$$\phi(u_1, ..., u_M) = (u_1, ..., u_m)$$

is a smooth one-to-one map of $U$ onto an open subset $X$ of $\mathbf{R}^m$. In $X$,

$$\phi^{-1}(u_1, ..., u_m) = (u_1, ..., u_m, g_{m+1}(u_1, ..., u_m), ..., g_M(u_1, ..., u_m)).$$

Hence, $\phi$ is a diffeomorphism (see Appendix A).

A subset of $\mathbf{R}^M$ is a smooth $m$-dimensional manifold if and only if it is locally diffeomorphic to $\mathbf{R}^m$. By extension, any set that is related to this subset by a bijective map is also a smooth $m$-dimensional manifold. Thus:

**PROPOSITION 1:** *The configuration space $\mathcal{C}$ of a rigid object $\mathcal{A}$ moving in an $N$-dimensional workspace is a smooth manifold of dimension $m = \frac{1}{2}N(N+1)$. The bijective application that maps any configuration of $\mathcal{C}$ to its representation in $\mathcal{M}_{\mathcal{C}} \subset \mathbf{R}^M$, with $M = N(N+1)$, is called an* **embedding** *of $\mathcal{C}$ in $\mathbf{R}^M$. $\mathbf{R}^M$ is called the* **ambient** *Euclidean space.*

That $\mathcal{C}$ is a manifold[3] means that its local topological and differential structures are identical to those of $\mathbf{R}^m$. It is directly related to the well-known fact that infinitesimal rotations (equivalently, angular velocities) form a vector space [Paul, 1981].

The local likeness of $\mathcal{C}$ and $\mathbf{R}^m$ is emphasized by the standard definitions given in the following subsection.

### 3.2 Charts and Atlas

Let $u : \mathcal{C} \rightarrow \mathcal{M}_{\mathcal{C}}$ be the bijective application that maps any configuration $\mathbf{q}$ in $\mathcal{C}$ to its representation $\mathbf{u}$ in $\mathcal{M}_{\mathcal{C}} = \mathbf{R}^N \times SO(N)$.

A pair $(U, \phi)$ consisting of an open subset $U$ of $\mathcal{C}$ and a diffeomorphism $\phi$ of the image of $U$ in $\mathcal{M}_{\mathcal{C}}$ onto $X$, an open subset of $\mathbf{R}^m$, is called a **chart** or a **coordinate system** on $U$. For any $\mathbf{q}$ in $U$, let us write $\phi(u(\mathbf{q})) = (x_1(u(\mathbf{q})), ..., x_m(u(\mathbf{q})))$. The $m$ smooth functions $x_1, ..., x_m$ are called the **coordinate functions** or more simply the **coordinates** of $\mathbf{q}$ in the chart. The inverse diffeomorphism $\phi^{-1}$ is called a **parameterization** of $U$. In the following, we will simply write $\phi(\mathbf{q}) = (x_1(\mathbf{q}), ..., x_m(\mathbf{q}))$.

Consider two charts $(U, \phi)$ and $(V, \psi)$ such that $U \cap V \neq \emptyset$ (see Figure 2). Let $X$ and $Y$ be the images of $U$ and $V$ by $\phi$ and $\psi$, respectively. The

[3] From now on, we will omit the word "smooth" and simply say that $\mathcal{C}$ is a manifold.

LATOMBE

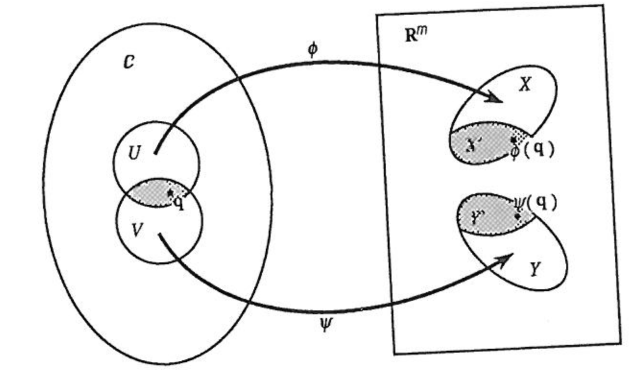


**Figure 2.** Two intersecting open subsets of $\mathcal{C}$, $U$ and $V$, are mapped diffeomorphically onto two open subsets $X$ and $Y$ of $\mathbf{R}^m$ by $\phi$ and $\psi$, respectively. $X'$ and $Y'$ denote the images of $U \cap V$ by $\phi$ and $\psi$. The composed applications $\psi \circ \phi^{-1} : X' \to Y'$ and $\phi \circ \psi^{-1} : Y' \to X'$ are smooth, and the charts $(U, \phi)$ and $(V, \psi)$ are said to be $C^\infty$-related.

intersection $U \cap V$ is open and thus has two different charts $(U \cap V, \phi)$ and $(U \cap V, \psi)$. Let $X' \subseteq X$ and $Y' \subseteq Y$ be the images of $U \cap V$ by $\phi$ and $\psi$, respectively. For every $\mathbf{q} \in U \cap V$, we write:

$$\phi(\mathbf{q}) = (x_1(\mathbf{q}), ..., x_m(\mathbf{q})),$$

$$\psi(\mathbf{q}) = (y_1(\mathbf{q}), ..., y_m(\mathbf{q})).$$

$\phi$ has a smooth inverse $\phi^{-1}$, so that the composed application $\psi \circ \phi^{-1}$ : $X' \to Y'$, with:

$$\psi \circ \phi^{-1}(x_1(\mathbf{q}), ..., x_m(\mathbf{q})) = (y_1(\mathbf{q}), ... y_m(\mathbf{q})),$$

is smooth. In the same way, the inverse application $\phi \circ \psi^{-1} : Y' \to X'$ is also smooth. The charts $(U, \phi)$ and $(V, \psi)$ are said to be $C^\infty$**-related.**

A set of charts $\{(U_\alpha, \phi_\alpha)\}$ whose domains $U_\alpha$ cover $\mathcal{C}$ is called an **atlas** of $\mathcal{C}$. Once an atlas has been defined for $\mathcal{C}$, we can represent $\mathcal{C}$ as a collection of copies of $\mathbf{R}^m$ without referring to its embedding in $\mathbf{R}^M$. Since $\mathbf{R}^N$ can be covered by a single chart, and $SO(N)$ is compact, $\mathcal{M}_\mathcal{C}$, hence $\mathcal{C}$, can be covered by a finite number of charts. (We recall that in a compact set $\mathcal{E}$, any infinite collection of open subsets covering $\mathcal{E}$ contains a finite subcollection covering $\mathcal{E}$.)

The fact that the charts in an atlas are $C^\infty$-related will allow us to extend the differential properties established in a chart of the atlas to all the other charts.

## 3.3 Other Embeddings

As we mentioned above, we can consider $\mathcal{C}$ as an abstract space independently of the ambient space $\mathbf{R}^M$. The latter then looks somewhat arbitrary. Indeed, we could now pick any space $\mathbf{R}^{M'}$, with $M' > M$, and easily represent $\mathcal{C}$ as a subset $\mathcal{M}'_\mathcal{C}$ of $\mathbf{R}^{M'}$ equivalent (i.e. globally diffeomorphic) to $\mathcal{M}_\mathcal{C}$. $\mathbf{R}^{M'}$ would then be another ambient space of $\mathcal{C}$.

Since $M$ is significantly greater than $m$, a legitimate question is the following: Does there exist $M'' < M$ such that $\mathbf{R}^{M''}$ contains a globally diffeomorphic copy $\mathcal{M}''_\mathcal{C}$ of $\mathcal{M}_\mathcal{C}$? A partial answer to this question is given by the Whitney Embedding Theorem: Every $r$-dimensional manifold can be embedded in $\mathbf{R}^{2r}$ [Guillemin and Pollack, 1974]. In other words, there is enough "twisting room" in $\mathbf{R}^{2r}$ for embedding any $r$-dimensional manifold. (Although the Whitney Theorem is optimal, it does not mean that a particular $r$-dimensional manifold cannot be embedded in a Euclidean space of dimension lower than $2r$; for instance, $\mathbf{R}^r$ is embedded in itself.)

Therefore, when we embed the one-dimensional manifold $SO(2)$ in $\mathbf{R}^4$ and the three-dimensional manifold $SO(3)$ in $\mathbf{R}^9$, there is some extra twisting room in the ambient spaces, which we do not use. We should at least be able to embed these manifolds in two- and six-dimensional Euclidean spaces, respectively.

Matrices in $SO(2)$ are of the form:

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} r_{11} & -r_{21} \\ r_{21} & r_{11} \end{pmatrix}.$$

$SO(2)$ can be mapped to the unit circle in $\mathbf{R}^2$, denoted by $S^1$, with the following mapping:

$$(r_{ij})_{i,j \in [1,2]} \in \mathbf{R}^4 \mapsto (r_{11}, r_{21}) \in \mathbf{R}^2.$$