APPENDIX E

ELEMENTARY DETECTION THEORY

(from class notes, ENSC 428)

5 FUNDAMENTALS — DETECTION OF ISOLATED PULSES

- The topics in this section comprise much of the core theory of digital communications. Many of them, and the way of thinking about the problems, will be useful to you in other contexts.

- You will learn:
  - a general representation of signals in a vector space with quadratic distance measure;
  - ways to formulate decision-making in statistical uncertainty
  - the structure of an optimum receiver for isolated pulses
  - the resulting probability of decision error

5.1 Signal Space $\boxed{\text{P+S 7.1}}$

- Many seemingly different mathematical entities and relations have similar structure. The ones we'll consider are:
  - vectors as arrays of real or complex numbers
  - vectors as square integrable functions
  - vectors as random variables
  - vectors as little arrows in Euclidean space

- A vector space consists of a set of vectors, denoted $\underline{x}$ or $\underline{y}$ that forms a commutative group under a binary operator "+"

$$\underline{x}+\underline{y}=\underline{y}+\underline{x} \qquad \underline{x}+(\underline{y}+\underline{z})=(\underline{x}+\underline{y})+\underline{z} \qquad \underline{x}+\underline{0}=\underline{x} \qquad \underline{x}+{}^{-}\underline{x}=\underline{0}$$

with a field of scalars and scalar multiplication

$$\alpha(\beta\underline{x})=(\alpha\beta)\underline{x} \qquad 1\underline{x}=\underline{x} \qquad 0\underline{x}=\underline{0} \qquad \alpha(\underline{x}+\underline{y})=\alpha\underline{x}+\alpha\underline{y}$$

$$(\alpha+\beta)\underline{x}=\alpha\underline{x}+\beta\underline{x}$$

Usually the scalars are in $\mathbb{R}$ or $\mathbb{C}$, but $\mathbb{B}=\{0,1\}$ will do, too

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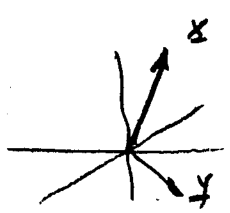
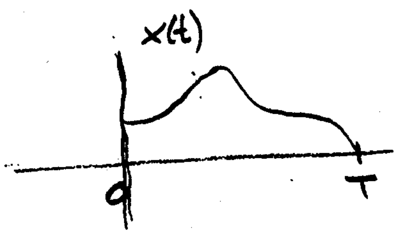
- Candidate vectors



little arrows in space

$$\underline{x}=\begin{bmatrix} 1 \\ 3-2j \\ \pi \\ j6 \end{bmatrix}$$

tuples of complex numbers (incl scalars)



functions in $L_2(T)$

x

$p_x(x)$

random variables

- We can add different vectors together. There's a zero element and there's a "negative" counterpart for every vector. Also, we can multiply by scalars.

- A vector space has no concept of length yet, or of distance between vectors.

- An inner product space is a vector space with an inner product operation $(\mathcal{V} \times \mathcal{V}) \to \mathbb{C}$ denoted $(\underline{x}, \underline{y})$:

$$(\underline{x}, \underline{y}) = (\underline{y}, \underline{x})^* \qquad (\alpha \underline{x} + \beta \underline{y}, \underline{z}) = \alpha (\underline{x}, \underline{z}) + \beta (\underline{x}, \underline{z})$$

$$(\underline{x}, \underline{x}) \geq 0, \text{ equality iff } \underline{x} = \underline{0}$$

Examples:

$$\underline{x} \cdot \underline{y}$$

$$\underline{y}^T \underline{x}$$

$$\int_0^T x(t)\, y(t)\, dt$$

and if complex

$$\underline{y}^\dagger \underline{x}$$

$$\int_0^T x(t)\, y^*(t)\, dt$$

- This lets us define the squared norm, or squared "length" of a vector:

$$\underline{x}^T\underline{x} = |\underline{x}|^2 \qquad \int x^2(t)\,dt$$

for complex

$$\underline{x}^+\underline{x} = |\underline{x}|^2 \qquad \int |x(t)|^2\,dt$$

- as well as squared "distance" between vectors:

$$|\underline{x}-\underline{y}|^2 = (\underline{x}-\underline{y})^T(\underline{x}-\underline{y})$$

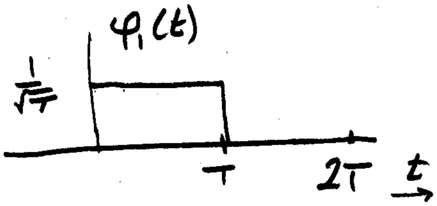
$$\int (x(t)-y(t))^2\,dt$$

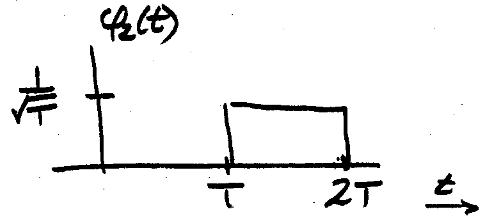
or for complex: $(\underline{x}-\underline{y})^+(\underline{x}-\underline{y}) \qquad \int |x(t)-y(t)|^2\,dt$

- Since waveforms with finite energy (square integrable) obey the same rules as $\mathbb{R}^n$ or $\mathbb{C}^n$, we can make vector diagrams of signals and use geometric intuition.

Example

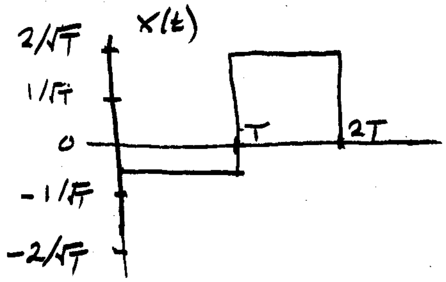
Assume these basis waveforms:

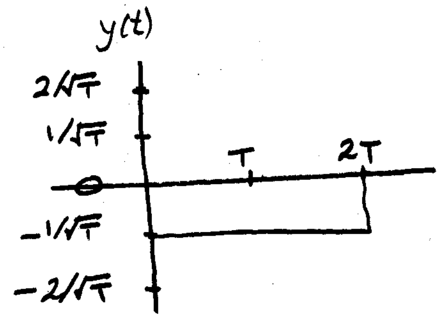




They have unit energy and are orthogonal (inner prod is zero) hence "orthonormal.

We can form other waveforms as linear combinations, such as

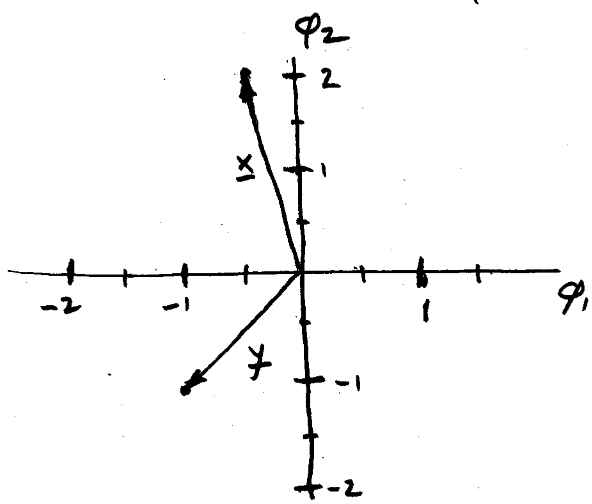




$x(t) = -\frac{1}{2}\varphi_1(t) + 2\varphi_2(t)$

$y(t) = -\varphi_1(t) - \varphi_2(t)$

so $\varphi_1(t)$, $\varphi_2(t)$ span the space of such linear combinations — they are a basis.



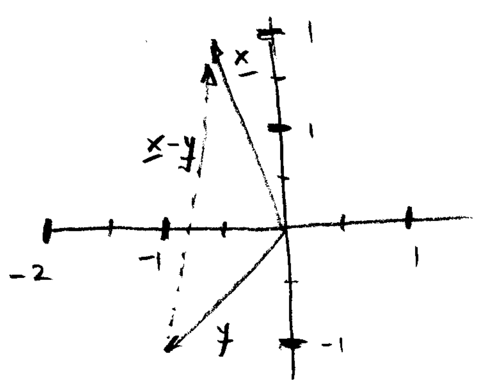
We can form sum and difference from diagram:

$\underline{x}(t) + \underline{y}(t)$ is represented by

Similarly, we can calculate the energy as the squared length:

Vector	Waveform
$\lvert \underline{y} \rvert^2 = \lvert -\underline{\varphi}_1 - \underline{\varphi}_2 \rvert^2$	$\mathcal{E}_y = \int_0^{2T} y^2(t)\,dt$
$= (-\underline{\varphi}_1 - \underline{\varphi}_2, -\underline{\varphi}_1 - \underline{\varphi}_2)$	$= \int_0^{2T} (-\varphi_1(t) - \varphi_2(t))(-\varphi_1(t) - \varphi_2(t))\,dt$
$= \lvert \underline{\varphi}_1 \rvert^2 + \lvert \underline{\varphi}_2 \rvert^2 = 2$	$= \int_0^{2T} \varphi_1^2(t)\,dt + \int_0^{2T} \varphi_2^2(t)\,dt$
or $(-1)^2 + (-1)^2 = 2$	$= 2$
$\lvert \underline{x} \rvert^2 = 2^2 + \left(\frac{1}{2}\right)^2 = 4\frac{1}{4}$	$\mathcal{E}_x = \int_0^{T} x^2(t)\,dt = \int_0^{T} (-\frac{1}{2}\varphi_1(t) + 2\varphi_2(t))^2\,dt$
	$= \frac{1}{4} + 4 = 4\frac{1}{4}$

and the "distance" between $x(t)$ and $y(t)$

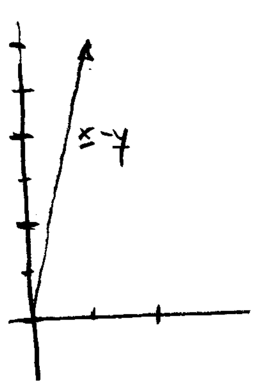


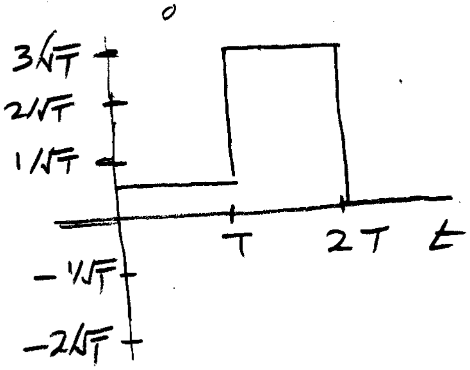
$$d^2 = \lvert \underline{x} - \underline{y} \rvert^2$$

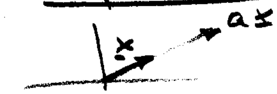
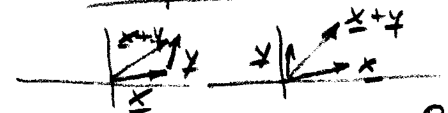
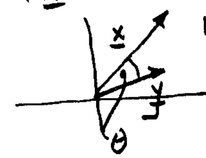
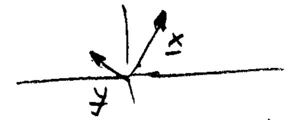
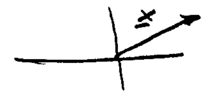
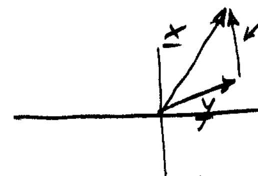
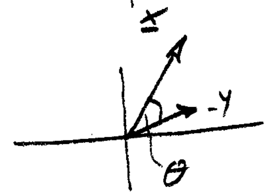
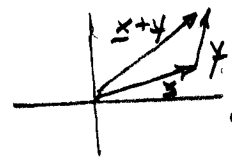
$$= \lvert (x_1 - y_1)\underline{\varphi}_1 + (x_2 - y_2)\underline{\varphi}_2 \rvert^2$$

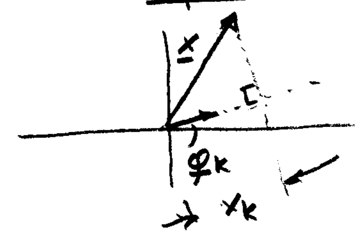
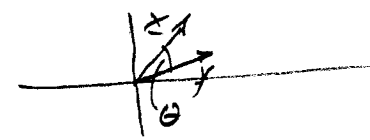
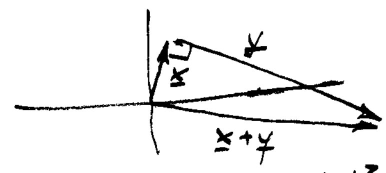
$$= \left(\frac{1}{2}\right)^2 + 3^2 = 9\frac{1}{4}$$

$$\int_0^{2T} (x(t) - y(t))^2\,dt = 9\frac{1}{4}$$





Concept	Denoted	$\mathcal{C}^n$	L_2	Random Vars	Space
scaling	$a\underline{x}$	$a\underline{x}$	$a\,x(t)$	ax	
sum	$\underline{x}+\underline{y}$	$\underline{x}+\underline{y}$	$x(t)+y(t)$	$x+y$	
inner prod	$(\underline{x},\underline{y})$	$\underline{y}^{+}\underline{x}$	$\int_0^T x(t)\,y^*(t)\,dt$	$\sigma_{xy}^2 = E[xy^*]$	 $\lvert\underline{x}\rvert\lvert\underline{y}\rvert\cos\Theta$
orthogonality	$(\underline{x},\underline{y})=0$	$\underline{y}^{+}\underline{x}=0$	$\int_0^T x(t)\,y^*(t)\,dt=0$	$\sigma_{xy}^2=0$	
norm	$\lVert\underline{x}\rVert$	$\sqrt{\underline{x}^{+}\underline{x}}=\lvert\underline{x}\rvert$	$\sqrt{\int_0^T \lvert x(t)\rvert^2 dt}=\sqrt{\mathcal{E}_x}$	σ_x	 length
distance	$d(\underline{x},\underline{y}) = \lVert\underline{x}-\underline{y}\rVert$	$\sqrt{(\underline{x}-\underline{y})^{+}(\underline{x}-\underline{y})} = \sum_i \lvert x_i-y_i\rvert^2$	$\sqrt{\int_0^T \lvert x(t)-y(t)\rvert^2 dt}$	$z=x-y$ σ_z	 $\underline{x}-\underline{y}$ Length
normalized inner prod	$\frac{(\underline{x},\underline{y})}{\lVert\underline{x}\rVert\,\lVert\underline{y}\rVert}$	$\frac{\underline{y}^{+}\underline{x}}{\sqrt{\underline{x}^{+}\underline{x}}\sqrt{\underline{y}^{+}\underline{y}}}$	$\frac{\int_0^T x(t)\,y^*(t)\,dt}{\sqrt{\int_0^T \lvert x(t)\rvert^2 dt \int_0^T \lvert y(t)\rvert^2 dt}}$	$\rho = \frac{\sigma_{xy}^2}{\sigma_x\sigma_y}$	 $\cos\Theta$
triangle $\neq$ty equality when $\underline{x}=a\underline{y}$	$\lVert\underline{x}+\underline{y}\rVert \le \lVert\underline{x}\rVert+\lVert\underline{y}\rVert$	$\lvert\underline{x}+\underline{y}\rvert \le \lvert\underline{x}\rvert+\lvert\underline{y}\rvert$	$\sqrt{\mathcal{E}_{x+y}} \le \sqrt{\mathcal{E}_x}+\sqrt{\mathcal{E}_y}$	$\sigma_{x+y}\le\sigma_x+\sigma_y$	 length ≤ sum of lengths other two sides

Concept	Denoted	$\mathcal{C}^n$	L_2	RandVars	Space
component projection $\underline{\varphi}_k$ unitnorm	$x_k = (\underline{x}, \underline{\varphi})$	$x_k = \underline{\varphi}_k^{+} \underline{x}$	$x_k = \int_0^T x(t)\varphi_k^*(t)\,dt$	$x_k = \sigma_{x\varphi}^2$	
Schwartz ≠ty equality when $\underline{x} = a\underline{y}$	$\lvert(\underline{x}, \underline{y})\rvert^2 \leq \lVert\underline{x}\rVert^2 \lVert\underline{y}\rVert^2$	$\lvert \underline{y}^{+} \underline{x}\rvert^2 \leq \lvert\underline{y}\rvert^2 \lvert\underline{x}\rvert^2$	$\left\lvert\int_0^T x(t) y^*(t)\,dt\right\rvert^2 \leq \mathcal{E}_x \mathcal{E}_y$	$\lvert\sigma_{xy}^2\rvert^2 \leq \sigma_x^2 \sigma_y^2$	 $\lvert\underline{x}\rvert^2 \lvert\underline{y}\rvert^2 \cos^2\theta \leq \lvert\underline{x}\rvert^2 \lvert\underline{y}\rvert^2$
Pythagoras for orthog	$\lVert\underline{x}+\underline{y}\rVert^2 = \lVert\underline{x}\rVert^2 + \lVert\underline{y}\rVert^2$ iff $(\underline{x}, \underline{y}) = 0$	$\lvert\underline{x}+\underline{y}\rvert^2 = \lvert\underline{x}\rvert^2 + \lvert\underline{y}\rvert^2$	$\mathcal{E}_{x+y} = \mathcal{E}_x + \mathcal{E}_y$ if orthog	$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$ if $\sigma_{xy}^2 = 0$	 $\lvert\underline{x}+\underline{y}\rvert^2 = \lvert\underline{x}\rvert^2 + \lvert\underline{y}\rvert^2$ since orthog