APPENDIX M

CHARACTERISTIC FUNCTION OF A GAUSSIAN QUADRATIC FORM

This is from the book M. Schwartz, WR Bennett and S. Stein, <u>Communication Systems and Techniques</u>, McGraw-Hill, 1968. Regrettably, it is no longer in print.

The last third of the book is a marvellous coverage of fading channels and what to do about them. New researchers in the area often find that for many, many topics, Seymour Stein was there first and better.

M. Schwartz, W Bennett and S Stein
Communication Systems and Techniques
McGraw Hill, 1966

APPENDIX B

DISTRIBUTION OF HERMITIAN QUADRATIC FORM IN COMPLEX GAUSSIAN VARIATES

1. Matrix Definitions

Let $\{z_i\}$, $i = 1, \ldots, M$ be M jointly distributed complex Gaussian variates, with mean values $\{\langle z_i\rangle\}$. Let these be arrayed in an $M \times 1$ column matrix

$$\mathbf{z} = [z_i] \tag{B-1-1}$$

Then the complex covariances among the $\{z_i\}$ are given by the elements R_{ik} of the $M \times M$ covariance matrix,

$$\mathbf{R} = \tfrac{1}{2}\langle[\mathbf{z} - \langle\mathbf{z}\rangle]^*[\mathbf{z} - \langle\mathbf{z}\rangle]^T\rangle \tag{B-1-2}$$

where $\mathbf{z}^T$ is the $1 \times M$ row matrix which is the transpose of $\mathbf{z}$. Note that while the multiplication in (B-1-2) defines an $M \times M$ matrix, multiplication in the opposite order defines a 1×1 matrix, a scalar, which is generally the squared magnitude of the complex vector involved. E.g., for a complex $M \times 1$ column vector $\mathbf{t}$,

$$|\mathbf{t}|^2 = \mathbf{t}^T\mathbf{t}^* = \mathbf{t}^{T*}\mathbf{t} = \sum_{i=1}^{M} |t_i|^2 \tag{B-1-3}$$

From (B-1-2) it is clear that

$$\mathbf{R}^T = \mathbf{R}^* \tag{B-1-4}$$

This identifies the covariance matrix as Hermitian.

For any matrix $\mathbf{F} = [F_{ik}]$, a quadratic form in complex variables $\{t_i\}$, which form the elements of a column vector $\mathbf{t}$, is defined by

$$f = \Sigma t_i^* F_{ik} t_k = \mathbf{t}^{T*}\mathbf{F}\mathbf{t} \tag{B-1-5}$$

Then $\mathbf{F}$ is said to be the matrix of the quadratic form. If $\mathbf{F}$ is Hermitian, it is readily shown that

$$f = f^* \tag{B-1-6}$$

that is, a Hermitian quadratic form is *real*. If, furthermore, $\mathbf{F}$ is such that $f > 0$ for arbitrary $\mathbf{t}$, both $\mathbf{F}$ and its quadratic form are said to be Hermitian *positive definite*; if $f \geq 0$ for arbitrary $\mathbf{t}$, they are said to be *positive semidefinite* (with similar definitions of negative definiteness). From the definition of (B-1-2), it is readily shown that any covariance matrix $\mathbf{R}$ is Hermitian, positive semidefinite. Furthermore, unless a deterministic linear relationship exists among some or all of the $\{z_i\}$, $\mathbf{R}$ will be Hermitian positive definite. The latter will be assumed to be the case below.

A standard result in matrix algebra is that an $M \times M$ Hermitian matrix has exactly

[1] The material here closely follows that outlined in G. L. Turin, The Characteristic Function of Hermitian Quadratic Forms in Complex Normal Variables, *Biometrika*, vol. 47, pp. 199–201, June, 1960.

M real (not necessarily distinct) eigenvalues $\{\lambda_i\}$. If the matrix is positive definite, its eigenvalues are positive real. The corresponding eigenvectors (column matrices) can be constructed to form an orthonormal set. Thus a unitary $M \times M$ matrix, $\mathbf{U}$, can be formed with the M eigenvectors of $\mathbf{R}$ as its columns, such that

$$\mathbf{U}^{T*}\mathbf{U} = \mathbf{I} \tag{B-1-7a}$$
$$\mathbf{U}^{T*}\mathbf{R}\mathbf{U} = \mathbf{\Lambda} \tag{B-1-7b}$$
$$\mathbf{R} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{T*} \tag{B-1-7c}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{\Lambda}$ is a diagonal matrix (all elements off the main diagonal are zero) with the M eigenvalues of $\mathbf{R}$ as its elements (in the same order in which the corresponding eigenvectors appear as columns of $\mathbf{U}$),

$$(\mathbf{\Lambda})_{ik} = \lambda_i\delta_{ik} \tag{B-1-8}$$

Since $\det \mathbf{U} = 1$, it follows from (B-1-7c) that

$$\det \mathbf{R} = \prod_{i=1}^{M} \lambda_i \tag{B-1-9}$$

That is, the determinant of any Hermitian matrix is equal to the product of its eigenvalues. It is also well known that the trace (sum of the diagonal components) of a Hermitian matrix is preserved under transformations such as (B-1-7b),

$$\sum_{i=1}^{M} R_{ii} = \sum_{i=1}^{M} \lambda_i \tag{B-1-10}$$

From (B-1-9), the statement that $\mathbf{R}$ is Hermitian positive definite is equivalent to the statement that $\det \mathbf{R} > 0$, and hence $\mathbf{R}$ is nonsingular. Thus there exists a well-defined inverse of $\mathbf{R}$, written as $\mathbf{R}^{-1}$. One form for writing the inverse, related to (B-1-7c), is simply

$$\mathbf{R}^{-1} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}^{T*} \tag{B-1-11}$$

where $\mathbf{\Lambda}^{-1}$ is the diagonal matrix with positive real (nonzero) diagonal elements $\{1/\lambda_i\}$. The inverse is obviously also Hermitian, positive definite. In addition, if $\mathbf{G}(\mathbf{R}) = \Sigma a_m\mathbf{R}^m$ is any matrix polynomial in $\mathbf{R}$, the eigenvalues of $\mathbf{G}$ are given as identical polynomials in the corresponding eigenvalues of $\mathbf{R}$,

$$g_i = \Sigma a_m\lambda_i^m \tag{B-1-12}$$

For example, the eigenvalues of $\mathbf{I}$-$\mathbf{R}$ are $\{1 - \lambda_i\}$.

2. Reduction to Diagonal Hermitian Form in Independent Variates

Let the M jointly distributed complex Gaussian variates $\{z_i\}$ defined by (B-1-1) and (B-1-2) be written in terms of their real and imaginary components,

$$z_i = x_i + jy_i \tag{B-2-1}$$

With the Hermitian positive definite nature of $\mathbf{R}$ as discussed above, it is then readily shown[1] that when the quadrature components have the statistical properties of the components of stationary bandpass noise, the joint p.d.f. of the M pairs of real Gaussian variates (x_i, y_i) can be written in the form

$$p(\{x_i,y_i\}) = \frac{1}{(2\pi)^M \det \mathbf{R}} \exp\left[-\tfrac{1}{2}(\mathbf{z} - \langle\mathbf{z}\rangle)^T\mathbf{R}^{-1}(\mathbf{z} - \langle\mathbf{z}\rangle)^*\right] \tag{B-2-2}$$

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$$p(\{x_i, y_i\}) = \frac{1}{(2\pi)^M \det \mathbf{R}} \exp\left[-\tfrac{1}{2}(\mathbf{z} - \langle\mathbf{z}\rangle)^T\mathbf{R}^{-1}(\mathbf{z} - \langle\mathbf{z}\rangle)^*\right] \tag{B-2-2}$$

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Note that the complex notation in the exponential of (B-2-2) is used only as an algebraic convenience which usefully summarizes the relationships implied among the $\{x_i, y_i\}$ by the fact of their representing quadrature components of stationary complex Gaussian processes.

The problem to be considered is that of finding the p.d.f., through its characteristic function, of a general Hermitian form in the $\{z_i\}$. Let this general form (not necessarily positive definite or semidefinite) be denoted by

$$f = \mathbf{z}^{T*}\mathbf{F}\mathbf{z} \tag{B-2-3}$$

The desired result is most easily obtained by first finding a matrix transformation which simultaneously diagonalizes **F** and **R**.

The matrix **U** defined in (B-1-7) immediately provides a transformation which diagonalizes **R**. That is, $\mathbf{U}^T\mathbf{z}$ defines a set of statistically independent Gaussian variates with diagonal covariance matrix $\mathbf{\Lambda}$. Moreover, because of the positive real nature of the elements of $\mathbf{\Lambda}$, there are an infinity of matrices which allow a factorization of $\mathbf{\Lambda}$ in the form

$$\mathbf{\Lambda} = \boldsymbol{\psi}^*\boldsymbol{\psi}^T \tag{B-2-4}$$

For example, one such factorization is the "square-root" matrix, in which $\boldsymbol{\psi}$ is the diagonal matrix with elements $\psi_i = \lambda_i^{1/2}$. However, if $\boldsymbol{\psi}$ is any solution to (B-2-4), then so is $\boldsymbol{\psi}\mathbf{V}$ where **V** is any arbitrary unitary matrix. The transformation

$$\mathbf{w} = \boldsymbol{\psi}^{-1}\mathbf{U}^T\mathbf{z} \tag{B-2-5a}$$

where $\boldsymbol{\psi}$ is any solution to (B-2-4) then further transforms the $\{z_i\}$ to a set $\{w_i\}$ which are statistically independent, and have unit variances. Thus,

$$\langle\mathbf{w}\rangle = \boldsymbol{\psi}^{-1}\mathbf{U}^T\langle\mathbf{z}\rangle \tag{B-2-5b}$$

and the covariance matrix of **w** is simply the identity matrix **I**. With its real and imaginary parts defined by

$$w_i = u_i + jv_i \tag{B-2-6}$$

the p.d.f. of these variates is simply

$$\begin{aligned} p(\{u_i,v_i\}) &= \frac{1}{(2\pi)^M} \exp\left[-\tfrac{1}{2}(\mathbf{w} - \langle\mathbf{w}\rangle)^T(\mathbf{w} - \langle\mathbf{w}\rangle)^*\right] \\ &= \frac{1}{(2\pi)^M} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{M} |w_i - \langle w_i\rangle|^2\right] \end{aligned} \tag{B-2-7}$$

The inverse of (B-2-5a) is

$$\mathbf{z} = \mathbf{U}^*\boldsymbol{\psi}\mathbf{w} \tag{B-2-8}$$

With this transformation, the quadratic form of (B-2-3) becomes

$$f = \mathbf{w}^{T*}(\boldsymbol{\psi}^{T*}\mathbf{U}^T\mathbf{F}\mathbf{U}^*\boldsymbol{\psi})\mathbf{w} = \mathbf{w}^{T*}\mathbf{T}\mathbf{w} \tag{B-2-9a}$$

where

$$\mathbf{T} = \boldsymbol{\psi}^{T*}\mathbf{U}^T\mathbf{F}\mathbf{U}^*\boldsymbol{\psi} \tag{B-2-9b}$$

is obviously also Hermitian. Thus f is now expressed as a Hermitian quadratic form in independent complex Gaussian variates. However **T** itself, being Hermitian, can also be diagonalized, in a form

$$\mathbf{T} = \mathbf{S}\mathbf{\Phi}\mathbf{S}^{T*} \tag{B-2-10}$$

where **S** is a unitary matrix of orthonormalized eigenvectors of **T**, and $\mathbf{\Phi}$ is the diagonal matrix of its eigenvalues, Φ_i (real, but not necessarily positive). Thus, one can introduce the transformation

$$\mathbf{n} = \mathbf{S}^{T*}\mathbf{w} \tag{B-2-11a}$$

$$\mathbf{w} = \mathbf{S}\mathbf{n} \tag{B-2-11b}$$

in terms of which the quadratic form is diagonal,

$$f = \mathbf{n}^{T*}\mathbf{\Phi}\mathbf{n} = \sum_{i=1}^{M} \Phi_i|\eta_i|^2 \tag{B-2-12}$$

Moreover, the covariance matrix of the $\{\eta_i\}$ is still **I**. Hence, with

$$\eta_i = \alpha_i + j\beta_i \tag{B-2-13}$$

$$\langle\mathbf{n}\rangle = \mathbf{S}^{T*}\boldsymbol{\psi}^{-1}\mathbf{U}^T\langle\mathbf{z}\rangle \tag{B-2-14}$$

and the relevant p.d.f. is

$$p(\{\alpha_i,\beta_i\}) = \frac{1}{(2\pi)^M} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{M} |\eta_i - \langle\eta_i\rangle|^2\right] = \prod_{i=1}^{M} p(\alpha_i,\beta_i) \tag{B-2-15}$$

where

$$p(\alpha_i,\beta_i) = \frac{1}{2\pi}\exp\left[-\tfrac{1}{2}|\eta_i - \langle\eta_i\rangle|^2\right] = \frac{1}{2\pi}\exp\left[-\frac{(\alpha_i - \langle\alpha_i\rangle)^2 + (\beta_i - \langle\beta_i\rangle)^2}{2}\right] \tag{B-2-16}$$

At the same time, the form of f in terms of these variates is simply

$$f = \sum_{i=1}^{M} \Phi_i(\alpha_i^2 + \beta_i^2) \tag{B-2-17}$$

3. Characteristic Function and P.D.F. of Hermitian Quadratic Form

The characteristic function of f, defined as a Fourier transform on its p.d.f., is

$$G_f(\xi) = \int_{-\infty}^{\infty} \exp(j\xi f)p(f)\,df = \langle\exp(j\xi f)\rangle \tag{B-3-1}$$

with its p.d.f. given as the inverse

$$p(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-j\xi f)G_f(\xi)\,d\xi \tag{B-3-2}$$

(If **F** were always Hermitian positive semidefinite, so that $f \geq 0$ for all **z**, a Laplace-transform characteristic function would be more convenient for subsequent interpretation; the distinction is however nonessential.) With (B-2-16) and (B-2-17), one obtains immediately the evaluation

$$G_f(\xi) = \left\langle \exp\left[j\xi\sum_{i=1}^{M}\Phi_i(\alpha_i^2 + \beta_i^2)\right]\right\rangle \tag{B-3-3}$$

$$= \prod_{i=1}^{M} \frac{1}{1 - 2j\xi\Phi_i}\exp\left[\left(\frac{j\xi\Phi_i}{1 - 2j\xi\Phi_i}\right)(\langle\alpha_i\rangle^2 + \langle\beta_i\rangle^2)\right]$$

$$= \frac{\exp\left[j\xi\sum_{i=1}^{M}(\Phi_i/1 - 2j\xi\Phi_i)|\langle\eta_i\rangle|^2\right]}{\prod_{i=1}^{M}(1 - 2j\xi\Phi_i)} \tag{B-3-4}$$

This result can be reformulated in terms of the original matrices $\mathbf{R}$ and $\mathbf{F}$. To this end, it is readily shown that the eigenvalues of $\mathbf{T}$ defined by (B-2-10) are also the eigenvalues of the (generally non-Hermitian) matrix

$$\mathbf{T}' = \mathbf{U}^*\boldsymbol{\psi}\mathbf{T}\boldsymbol{\psi}^{-1}\mathbf{U}^T = \mathbf{U}^*\boldsymbol{\psi}\boldsymbol{\psi}^{T*}\mathbf{U}^T\mathbf{F} = \mathbf{R}^*\mathbf{F} \tag{B-3-5}$$

Next, from (B-1-12) and (B-1-9), the denominator in (B-3-4) can be written as

$$\prod_{i=1}^{M} (1 - 2j\xi\Phi_i) = \det [\mathbf{I} - 2j\xi\mathbf{T}] \tag{B-3-6}$$

But also $\det \boldsymbol{\psi}^{-1} = [\det \boldsymbol{\psi}]^{-1}$, and $\det \mathbf{U} = 1$. Thus, with (B-3-5), one can further write

$$\begin{aligned}\prod_{i=1}^{M} (1 - 2j\xi\Phi_i) &= \det [\mathbf{U}^*\boldsymbol{\psi}(\mathbf{I} - 2j\xi\mathbf{T})\boldsymbol{\psi}^{-1}\mathbf{U}^T] \\ &= \det [\mathbf{I} - 2j\xi\mathbf{T}'] = \det [\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F}]\end{aligned} \tag{B-3-7}$$

For the exponential in (B-3-4), one can recognize that $1/(1 - 2j\xi\Phi_i)$ are the elements of the diagonal matrix $(\mathbf{I} - 2j\xi\boldsymbol{\Phi})^{-1}$. One can thus write, as alternative possibilities,

$$\exp \left[j\xi \sum_{i=1}^{M} \frac{\Phi_i}{1 - 2j\xi\Phi_i} |\langle \eta_i \rangle|^2 \right] = \exp [j\xi\langle \mathbf{n} \rangle^{T*}\boldsymbol{\Phi}(\mathbf{I} - 2j\xi\boldsymbol{\Phi})^{-1}\langle \mathbf{n} \rangle] \tag{B-3-8}$$

$$= \exp \left[-\frac{1}{2} \sum_{i=1}^{M} \left(1 - \frac{1}{1 - 2j\xi\Phi_i} \right) |\langle \eta_i \rangle|^2 \right] = \exp \{ -\tfrac{1}{2}\langle \mathbf{n} \rangle^{T*}[\mathbf{I} - (\mathbf{I} - 2j\xi\boldsymbol{\Phi})^{-1}]\langle \mathbf{n} \rangle \} \tag{B-3-9}$$

By inserting the definition of $\langle \mathbf{n} \rangle$ from (B-2-14), and utilizing the other matrix definitions in (B-2-10), (B-2-4), and (B-1-11), one finds the second of these to be equivalent to either of the forms[1]

$$\exp \{ -\tfrac{1}{2}\langle \mathbf{n} \rangle^{T*}[\mathbf{I} - (\mathbf{I} - 2j\xi\boldsymbol{\Phi})^{-1}]\langle \mathbf{n} \rangle \}$$
$$= \exp \{ -\tfrac{1}{2}\langle \mathbf{z} \rangle^{T*}(\mathbf{R}^*)^{-1}[\mathbf{I} - (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})^{-1}]\langle \mathbf{z} \rangle \} \tag{B-3-10a}$$
$$= \exp \{ -\tfrac{1}{2}\langle \mathbf{z} \rangle^{T*}[\mathbf{I} - (\mathbf{I} - 2j\xi\mathbf{F}\mathbf{R}^*)^{-1}](\mathbf{R}^*)^{-1}\langle \mathbf{z} \rangle \} \tag{B-3-10b}$$

Likewise, from (B-3-8) one finds the possibly more useful form

$$\exp [j\xi\langle \mathbf{n} \rangle^{T*}\boldsymbol{\Phi}(\mathbf{I} - 2j\xi\boldsymbol{\Phi})^{-1}\langle \mathbf{n} \rangle] = \exp [j\xi\langle \mathbf{z} \rangle^{T*}(\mathbf{F}^{-1} - 2j\xi\mathbf{R}^*)^{-1}\langle \mathbf{z} \rangle] \tag{B-3-11}$$

The two results can also be related by utilizing the algebraic identity

$$\mathbf{I} = (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})(\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})^{-1} \tag{B-3-12}$$

With (B-3-7), and (B-3-10a) or (B-3-11), respectively, the characteristic function of (B-3-4) can be written in the respective forms

$$G_f(\xi) = \frac{\exp \{ -\tfrac{1}{2}\langle \mathbf{z} \rangle^{T*}(\mathbf{R}^*)^{-1}[\mathbf{I} - (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})^{-1}]\langle \mathbf{z} \rangle \}}{\det (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})} \tag{B-3-13}$$

$$= \frac{\exp [j\xi\langle \mathbf{z} \rangle^{T*}(\mathbf{F}^{-1} - 2j\xi\mathbf{R}^*)^{-1}\langle \mathbf{z} \rangle]}{\det (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})} \tag{B-3-14}$$

The Fourier-transform inversion which gives the p.d.f. from these results is given by (B-3-2).

[1] The forms in (B-3-10) coincide with those presented by Turin, *loc. cit.*

In the important case of a Hermitian form in a zero-mean complex Gaussian process,

$$\langle \mathbf{z} \rangle = 0 \tag{B-3-15}$$

the results in (B-3-13) or (B-3-14) reduce to the much simpler form

$$G_f(\xi) = \frac{1}{\det (\mathbf{I} - 2j\xi\mathbf{R}^*\mathbf{F})} \tag{B-3-16}$$

It was earlier noted parenthetically that when $\mathbf{F}$ is Hermitian *and* positive semidefinite, it may sometimes be more convenient to define the characteristic function as a Laplace transform on the p.d.f. Thus, rather than (B-3-1) and (B-3-2), one would have

$$G_f(s) = \int_0^{\infty} \exp (-sf)p(f)\, df = \langle \exp (-sf) \rangle \tag{B-3-17}$$

and the inversion

$$p(f) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \exp (sf) G_f(s)\, ds \tag{B-3-18}$$

By comparing with the derivations above, it is clear that the results in this case will be the respective forms

$$G_f(s) = \frac{\exp \{ -\tfrac{1}{2}\langle \mathbf{z} \rangle^{T*}(\mathbf{R}^*)^{-1}[\mathbf{I} - (\mathbf{I} + 2s\mathbf{R}^*\mathbf{F})^{-1}]\langle \mathbf{z} \rangle \}}{\det (\mathbf{I} + 2s\mathbf{R}^*\mathbf{F})} \tag{B-3-19}$$

$$= \frac{\exp [-s\langle \mathbf{z} \rangle^{T*}(\mathbf{F}^{-1} + 2s\mathbf{R}^*)^{-1}\langle \mathbf{z} \rangle]}{\det (\mathbf{I} + 2s\mathbf{R}^*\mathbf{F})} \tag{B-3-20}$$

and in the case $\langle \mathbf{z} \rangle = 0$,

$$G_f(s) = \frac{1}{\det (\mathbf{I} + 2s\mathbf{R}^*\mathbf{F})} \tag{B-3-21}$$

APPENDIX C: CALCULATION OF THE MARCUM Q-FUNCTION

Thanks to Prof. John Bird, of the School of Engineering Science, Simon Fraser University, for this method of calculating Marcum's Q-function. The derivation can be found in J.S. Bird and D.A.George, "The Use of the Fourier-Bessel Series in Calculating Error Probabilities for Digital Communication Systems", *IEEE Trans Commun*, vol COM-29, no 9, pp 1357-1365, September 1981.

First, a procedure for the *k*th zero of $J_0(x)$. The first 30 were originally calculated with a root finder. The rest were obtained by McMahon's expansion for large zeros [Abromowitz and Stegun, *Handbook of Mathematical Functions*, formula 9.5.12]. Here the first 100 zeros are in an array, and McMahon's expansion is used only for indexes over 100.

read the first 100 zeros from a file:

first100 := READPRN("first100.txt")

$N := \mathrm{length}(\mathrm{first100}) \qquad N = 100$

McMahon's expansion:

$$b(k) := (k - 0.25)\cdot\pi \qquad bi(k) := \frac{0.125}{b(k)} \qquad bq(k) := bi(k)^2$$

$$\mathrm{J0zeropoly}(k) := b(k) + bi(k)\cdot\left[1 - bq(k)\cdot\left[\frac{124}{3} - bq(k)\cdot(8061.86666666666 - bq(k)\cdot 3826125.409523809)\right]\right]$$

$$z_J(k) := \mathrm{if}\left(k \le N, \mathrm{first100}_{k-1}, \mathrm{J0zeropoly}(k)\right)$$

Next, the calculation of the Q-function itself. It depends on two parameters: the number of terms in the series *M*, and the radius *R*. Guidelines in Bird and George say that if *R* is approximately *a*+10, then the error is negligible. Further, Figure 6 of the same reference shows that *M* should be about 25+3*a*

$$k_J := 1 .. 500$$

$$Q_{mc}(a,b) := 2\cdot\frac{b}{a+10}\cdot\sum_{k_J}\left(k_J < 25 + 3\cdot a\right)\cdot\frac{\exp\left[-\frac{1}{2}\cdot\left(\frac{z_J(k_J)}{a+10}\right)^2\right]}{z_J(k_J)\cdot J1\left(z_J(k_J)\right)^2}\cdot J0\left(\frac{a\cdot z_J(k_J)}{a+10}\right)\cdot J1\left(\frac{b\cdot z_J(k_J)}{a+10}\right)$$

$$Q_m(a,b) := 1 - Q_{mc}(a,b)$$

also see [Cant 87] PE Cantrell and AK Ojha, "Comparison of Generalized Q-function Algorithms," IEEE Trans Inf Th, vol IT-33, No 4, pp 591-596, July 87