Appendix N

One-Sided and Two-Sided Laplace Transforms

1. Introduction

You probably have some familiarity with the standard Laplace transform and its inversion by partial fractions. However, you may not have seen its links with the classical theory of complex variables and the role of residues in the inversion of a transform. These notes will fill a little of that gap. They're not a substitute for proper study of complex variables, but they will show you how to calculate transforms and invert them by resides (at least for the class of rational polynomial transforms).

Why the emphasis on residues? Because they are the easiest way to deal with the second major topic of these notes: the two-sided Laplace tranform.

Many functions in communications and signal processing are two-sided; that is, they are not necessarily zero for negative time. These notes show you how to represent them with the two-sided Laplace transform and why you also need to specify a "region of convergence". You will see how to invert two-sided transforms of rational polynomial type by residues.

2. Definition of One-Sided Transform

A one-sided function is zero for negative time; that is, $t < 0^-$, where 0^- denotes a time just before 0 (the formulation makes allowance for impulses at time zero, $d(t)$. For exponential, sinusoidal and polynomial signals, and for sytems described by linear differenial equations with constant coefficients, the laplace transform provides a convenient simplification. It's a way of expressing any function as a superposition (integral) of complex exponentials.

The Laplace transform of a function $x(t)$ is

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

where s is a complex variable. The inverse transform is given by

$$x(t) = \frac{1}{2pj} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} dt$$

where c is a real constant selected for convergence of $X(s)$, as discussed in detail in the next section.

A Short Table of One-Sided Transform Pairs	
$x(t),\ t \ge 0^-$	$X(s)$
e^{at}, real a	$\frac{1}{s-a}, \quad \mathrm{Re}[s] > a$
e^{pt}, real a	$\frac{1}{s-p}, \quad \mathrm{Re}[s] > \mathrm{Re}[p]$
$\sin(w\,t)$	$\frac{w}{s^2+w^2}, \quad \mathrm{Re}[s] > 0$
$\cos(w\,t)$	$\frac{s}{s^2+w^2}, \quad \mathrm{Re}[s] > 0$
$u_0(t), d(t)$	1
$u_{-1}(t)$	$\frac{1}{s}$
$u_{-2}(t)$	$\frac{1}{s^2}$

Questions:

1. Use the definition of the transform to verify the first entry in the table above.

2. What is the Laplace transform of the function

   $$x(t) = te^{-t}$$

   including the condition on $\mathrm{Re}[s]$?

3. The Laplace Transform on the Complex Plane

3.1 Its Origins

A function can be expanded on an interval $[0,T]$ as a Fourier series – a sum of sines and cosines with frequencies $w_k = k2p\,/T$ or, equivalently, as a sum of complex exponentials – as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jw_k t}, \quad 0 \le t < T$$

where

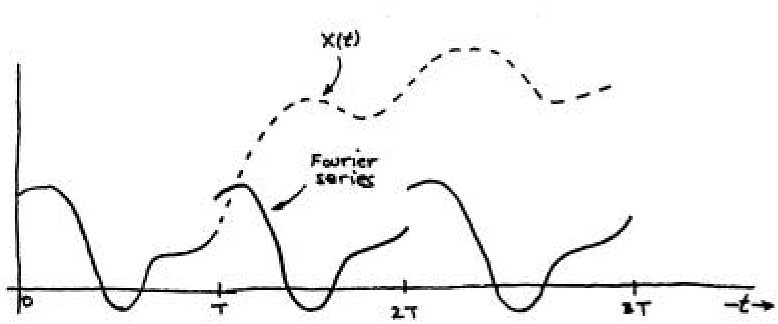
$$X_k = \frac{1}{T}\int_0^T x(t)\, e^{-jw_k t} dt\,, \quad w_k = k2p\,/\,T$$

since

$$\frac{1}{T}\int_0^T x(t)\, e^{-jw_k t} dt \;= \sum_{n=-\infty}^{\infty} \int_0^T X_n e^{jw_n t} e^{-jw_k t} dt = T\, X_k$$

The series is a resolution of $x(t)$ into components at frequencies w_k, each with complex amplitude X_k.

Although the resulting series is periodic, it does represent the function on [0,T], as shown in the sketch.



Now let us increase T, the length of the interval over which the series equals the original function. The frequencies w_k will become more closely spaced, and in the limit the sum becomes an integral

$$x(t) = \frac{1}{2p}\int_{-\infty}^{\infty} X(w)\, e^{jwt}\, dw$$

where

$$X(w) = \int_{0^-}^{\infty} x(t)\, e^{-jwt}\, dt$$

This type of resolution into components along a continuous frequency exis is known as the Fourier transform and the pair of integrals is known as the Fourier transform pair. The component of $x(t)$ at frequency w, $X(w)$, can be considered a density: if the units of $x(t)$ are volts, then the units of $X(w)$ are volt-sec (or volt/Hz if we had been using Hz instead of angular frequency in radian/sec). The transform $X(w)$ is complex, to represent both magnitude and phase.

When one attempts to calculate the Fourier tranform of some functions, however, the integral does not converge. Examples are the unit step $x(t) = u_{-1}(t)$, or $\sin(w_o t)$ or $\exp(at)$ for a>0 (try them and see). A way around this difficulty is first to multiply $x(t)$ by $e^{-s\,t}$, for some real $s > 0$, and Fourier transform the result. With s large enough, most useful functions will converge. We can easily regain $x(t)$ by multiplying the inverse transform by $e^{s\,t}$. The pair of equations now becomes

$$X(w) = \int_{0^-}^{\infty} x(t) e^{-(s+jw)t}\, dt$$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(s) e^{(s+jw)t}\, dw$$

Finally, a change of variable gives the Laplace tranform. Define the complex frequency variable $s = s + jw$. Then

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st}\, dt$$

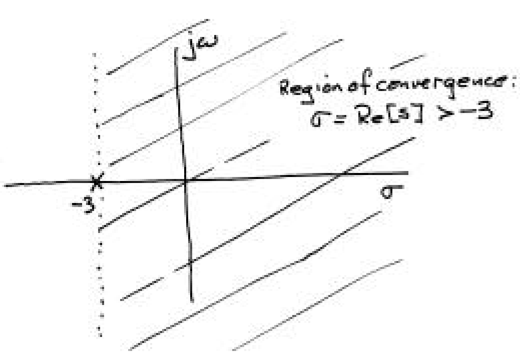
$$x(t) = \frac{1}{2pj} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds$$

where we pick $c = s$ to be large enough for convergence of the transform.

The values of s for which the real part s is large enough make up the "region of convergence". Consider, for example, transforming $x(t) = \exp(-3t)$:

$$X(s) = \int_{0^-}^{\infty} e^{-3t}\, e^{-st}\, dt = \int_{0^-}^{\infty} e^{-(3+s)t}\, dt = \frac{1}{s+3}, \quad \mathrm{Re}[s] > -3$$

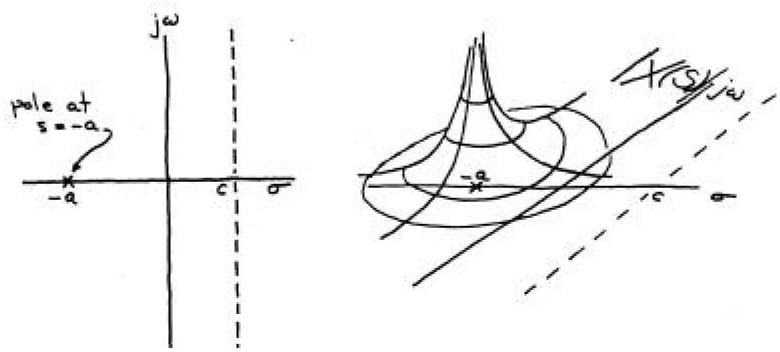
The pole and region of convergence are illustrated below:



3.2 Inversion by Line and Contour Integrals

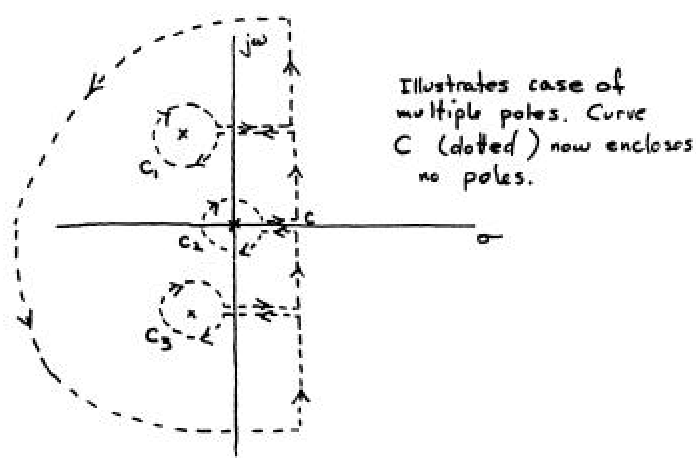
A Laplace transform $X(s)$ is apparently only going to be evaluated at points along the line $s = c + jw$ in the inverse transform integral. Nevertheless, $X(s)$ is a fullblown complex function of a complex variable $s = s + jw$, which can represent any point on the complex plane.

Consider the transform pair $x(t) = e^{-at}$ and $X(s) = 1/(s + a)$. The two sketches show clearly that the inverse transform is a line integral on the complex plane.



The region of convergence consists of all points to the right of s=-a – and generally, for one-sided tranforms, it is to the right of the rightmost pole – and the integration line must lie in the region of convergence. That is, c>-a.

The line integral suggests a contour integration. Consider the integration path shown below. Curve C consists of the original line integral, circles around the poles, and a giant semicircle to the left. The cuts do not have to be considered, since the integration in one direction is cancelled by the contribution from the opposite direction, which has opposite sign.



Now for some complex variable theory. First, the Cauchy Integral Theorem: if a function $F(s)$ is analytic within and on a closed curve C then

$$\oint_C F(s)\,ds = 0$$

This is certainly true of rational polynomial transforms. The practical result, demonstrated by the sketch below, is that our line integral equals the sum of integrals around each of the poles, less the integral along the semicircle.

Next consider the integration along the semicircle. If the integrand in

$$x(t) = \frac{1}{2pj}\int X(s)e^{st}\,ds$$

approaches zero more quickly than 1/|s| on that semicircle, then the contribution from the semicircle is negligible, and approaches zero as the radius increases. This condition is ensured by the combination of (1) positive t and large negative s in the exponential factor and (2) $|X(s)| \to 0$ uniformly as the radius becomes large. The latter condition is in turn guaranteed if the degree of the numerator of $X(s)$ is less than the degree of the denominator.

We now have the important result that our line integral equals the sum of integrals around each of the poles, so that

$$x(t) = \sum_i \oint_{C_i} \frac{1}{2p\,j} \quad X(s) \quad e^{st} \quad ds$$

where C_i is a closed curve encircling the i^{th} pole. Next, we use the Cauchy Integral Formula: if $F(s)$ is analytic within and on a closed curve C_i, and the point $s=a$ is within C_i, then

$$\frac{1}{2pj}\oint_{C_i}\frac{F(s)}{s-a}ds = F(a)$$

$F(a)$ is the "residue" of the integrand $F(s)/(s\text{-}a)$ when it is integrated around the curve centred on $s=a$. This is an extremely useful result. For example, we might want the inverse transform x(t) of $X(s) = \dfrac{1}{s+r}$. Then, as shown by the sketch above, we must integrate $e^{st}/(s+r)$ along a closed curve about the pole $s=-r$. In order to use the Cauchy Integral Formula, we identify $F(s)$ with e^{st} and a with $-r$. The inverse x(t) is obtained in one step: $x(t) = e^{-rt}$, the residue at $s=-r$, which agrees with the result obtained in Section 1.

To obtain the residue at a pole with multiplicity greater than one, note that differentiation of the Cauchy Integral Formula with respect to a yields

$$\frac{1}{n!}\frac{d^n F(s)}{ds^n}\bigg|_{s=a} = \frac{1}{2pj}\oint_{C_i} \frac{F(s)}{(s-a)^{n+1}}\,ds$$

which again lets us evaluate the residue without explicitly performing the integration

4. Inversion by Residues

In this section, we'll turn the contour integration and residue theory into a relatively mechanical procedure for inverting a Laplace transform of the rational polynomial type.

Given a Laplace transform $X(s)$, we want the associated inverse transform $x(t)$. The integrand in the inverse transform equation is then $X(s)e^{st}$. Define a *pole* of the integrand as a singularity point at which it becomes infinite (a loose definition, but it works for polynomials). For our rational polynomials, poles are roots of the denominator. For example, the integrand

$$\frac{(s^2+7)e^{st}}{(s-2)(s+3)^2(s^2+2s+5)}$$

has three simple poles, at s=2 and $s=-1\pm j2$, and a double pole at s=-3. Let there be N distinct poles s_k, k=1,…,N, each of multiplicity n_k. In our example, N=4: s_1=2, s_2=-3, s_3=-1+j2, s_4=-1-j2 and n_1=1, n_2=2, n_3=1 and n_4=1.

Next, define the *pole coefficient* associated with pole s_k as

$$F_k(s) = (s-s_k)^{n_k} X(s)\,e^{st}$$

In our example,

$$F_2(s) = \frac{(s^2+7)e^{st}}{(s-2)(s^2+2s+5)}$$

Notice that this definition simply removes the offending pole.

Finally, we just operate on the pole coefficients to produce the residue at pole s_k. For a simple pole the residue $R_k(t)$ is

$$R_k(t) = F_k(s_k)$$

In our example,

$$R_1(t) = \frac{11e^{2t}}{25 \cdot 13} = 0.0339e^{2t}$$

For a multiple pole, define the residue

$$R_k(t) = \frac{1}{(n_k - 1)!} \frac{d^{n_k - 1} F_k(s)}{ds^{n_k - 1}}\bigg|_{s=s_k}$$

which requires calculation of the $(n_k$-1)th derivative. This definition is clearly valid for simple poles, too. In our example,

$$R_2(t) = -\frac{2}{5}\left(\frac{13}{40} + t\right)e^{-3t}$$

after differentiation, substitution and simplification.

The inverse transform is, at last, just the sum of the residues:

$$x(t) = \sum_{k=1}^{N} R_k(t)$$

In our example,

$$x(t) = \frac{11e^{2t}}{25 \cdot 13} - \frac{2}{5}\left(\frac{13}{40} + t\right)e^{-3t} + \frac{e^{-t}}{4 \cdot 13}(5\cos(2t) + \sin(2t))$$

where the bracketed quantity in the last term can also be written as $\sqrt{26}\cos\left(2t + \tan^{-1}(1/5)\right)$.

<u>Questions</u>

3. Identify the poles and zeroes and their multiplicity in the transforms below:

(a) $X(s) = \dfrac{s+3}{s^3 + s^2 + 8s - 10}$

(b) $X(s) = \dfrac{s^2}{s^3 + 7s^2 + 15s + 9}$

and show their locations on the s-plane.

4. Invert the following transforms:

(a) $X(s)=\dfrac{2}{s+3}$

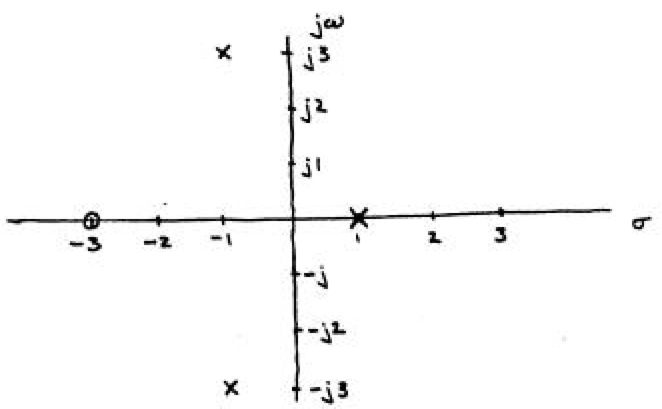
(b) $X(s)=\dfrac{s}{s^2-5s+6}$

(c) $X(s)=\dfrac{1}{s^2+2s+5}$

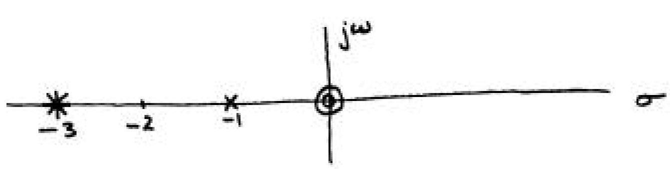
(d) $X(s)=\dfrac{s}{s^2+8s+16}$

<u>Answers</u>

3. (a) Poles at $s=-1\pm j3$, zero at $s=-3$, all simple.



(b) Poles at $s=1$, $s=-3$ (double), zero at $s=0$ (double)



4. (a) $x(t)=2e^{-3t},\quad t\ge 0$ single pole

(b) $x(t)=3e^{3t}-2e^{2t},\quad t\ge 0$ two simple poles

(c) $x(t)=\dfrac{1}{2}e^{-t}\sin(2t),\quad t\ge 0$ complex conjugate poles

(d) $x(t)=(1-4t)e^{-4t},\quad t\ge 0$ double pole

5. The Two-Sided Laplace Transform

5.1 Definition and Regions of Convergence

The two-sided Laplace transform allows time functions to be non-zero for negative time. It includes the one-sided transform that we have discussed already as a special case. The definition

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$$

seems straightforward enough, but there are some subtleties, which we discover through a set of three examples.

First, consider the function

$$x_1(t) = \begin{cases} 0, & t < 0 \\ e^{2t} - e^{-3t}, & t \geq 0 \end{cases}$$

It is strictly causal; that is, it is zero for negative time. Its transform is easily evaluated as

$$X_1(s) = \frac{5}{s^2 + s - 6}$$

with region of convergence (ROC) defined by $s > -3$ *and* $s > 2$ for the two terms, or simply $s > 2$. Next consider the two sided function

$$x_2(t) = \begin{cases} -e^{2t}, & t < 0 \\ -e^{-3t}, & t \geq 0 \end{cases}$$

Its transform is obtained by integration

$$\begin{aligned} X_2(s) &= -\int_{-\infty}^{0} e^{2t}\, e^{-st}\, dt - \int_{0}^{\infty} e^{-3t}\, e^{-st}\, dt \\ &= \underbrace{\frac{1}{s-2}}_{s<2} - \underbrace{\frac{1}{s+3}}_{s>-3} = \frac{5}{s^2 + s - 6} \end{aligned}$$

The region of convergence is defined by $s > -3$ (for the second, positive time, term) *and* $s < 2$ (for the first, negative time, term), or simply $-3 < s < 2$. Interesting – its transform is the same as $X_1(s)$. Finally, consider the strictly anticausal function

$$x_3(t) = \begin{cases} -e^{2t} + e^{-3t}, \ t < 0 \\ 0, \ t \geq 0 \end{cases}$$

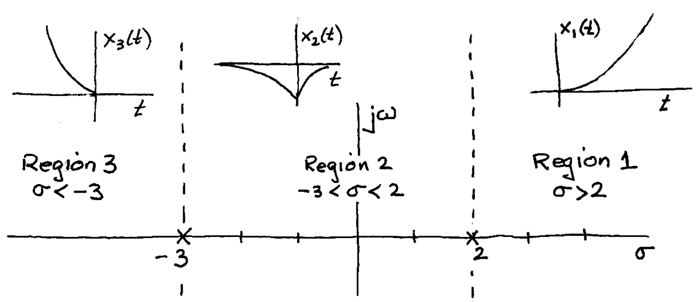
Its transform is given by

$$X_3(s) = -\int_{-\infty}^{0} e^{2t} \ e^{-st} \ dt \ + \ \int_{-\infty}^{0} e^{-3t} \ e^{-st} \ dt$$

$$= \underbrace{\frac{1}{s-2}}_{s<2} - \underbrace{\frac{1}{s+3}}_{s<-3} = \frac{5}{s^2+s-6}$$

and its region of convergence is defined by $s < 2$ (for the first term) *and* $s < -3$ (for the second term), or simply $s < -3$. Of more interest is that its transform is also equal to the other two: $X_1(s) = X_2(s) = X_3(s)$.

You have just seen three different time functions produce the same two-sided Laplace transform. Evidently, the transform alone is not sufficient to specify the time function – you need the transform and a region of convergence. Each such region is bounded by a pole on either side (except, of course, for the semi-infinite regions at the left and right), and each corresponds to a different time function, or inverse transform. The sketch below illustrates our example.



You may be wondering about region 2 in our example. It was convenient that the intersection of the two elementary regions of convergence $s > -3$ and $s > 2$ was not empty, giving $-3 < s < 2$. But are there cases in which the intersection is empty? Yes, although they are not often encountered. Consider, for example,

$$x_4(t) = \begin{cases} e^{-3t}, \ t < 0 \\ e^{2t}, \ t \geq 0 \end{cases}$$

which requires $s < -3$ *and* $s > 2$. Here, there is no region of the *s*-plane in which a single transform can represent all of $x_4(t)$. We can still handle this – we just use separate transforms for the negative time and positive time halves. Clumsy, but there aren't many options. We then have a pair of transforms

$$X_{4-}(s) = -\frac{1}{s+3}, \quad s < -3 \quad \text{for the negative time half}$$

$$X_{4+}(s) = \frac{1}{s-2}, \quad s > 2 \quad \text{for the positive time half}$$

<u>Questions</u>

5. Transform the function

$$x(t) = \begin{cases} -3e^{2t}, & t < 0 \\ -2e^{t}, & t \ge 0 \end{cases}$$

and sketch its pole-zero diagram with region of convergence.

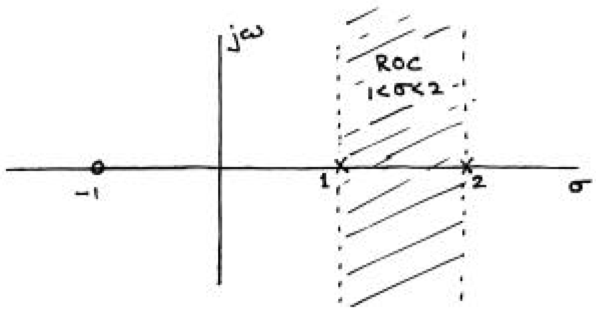
6. Consider a specific region of convergence for a Laplace transform $X(s)$. Were the poles to its right contributed by the negative time or positive time half of $x(t)$? How about the poles to its left? What can we conclude if it is an end region, e.g., no poles to its right?

7. Under what condition on region of convergence does the time function x(t) decay to zero on both sides of the origin, i.e., for $t \to \infty$ and for $t \to -\infty$?

8. If you convolve two time functions, the Laplace transform of the result is the product of the individual transforms. How is the ROC of the product related to the two original ROCs?

<u>Answers</u>

5. Transformation gives $X(s) = \dfrac{s+1}{s^2 - 3s + 2}$, with $1 < s < 2$.



6. Transformation of the positive time half of a function gives rise to constraints that s be greater than some values. Therefore, the poles to the left of the ROC are associated with positive time. Similarly, for negative time the constraints are that s be less than some value and the corresponding poles are to the right of the ROC.

7. For the function to decay to zero for positive time, the poles must be less than zero. These are the ones to the left of the ROC. Similarly, for the function to decay to zero for positive time, the poles must be greater than zero, and they are to the right of the ROC. Consequently, the ROC must include the origin (i.e., the imaginary axis) for the function to decay on both sides. This is not surprising, since the Laplace transform equals the Fourier transform on the imaginary axis, and convergence of the Fourier transform requires the decay on both sides.

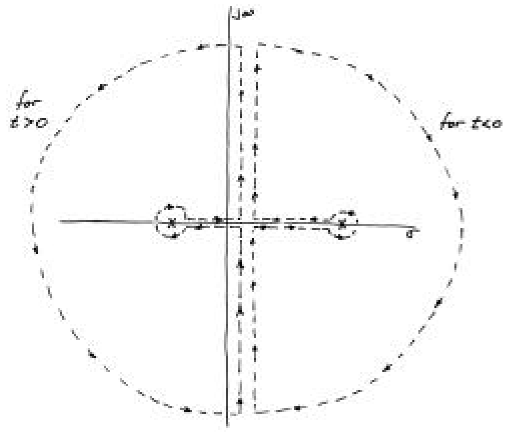
8. The ROC is the intersection of the two original ROCs. If the intersection is empty, then the convolution itself is unbounded and undefined.

5.2 Inversion of Two-Sided Transforms

If you can invert one-sided transforms by residues, then you can invert two-sided transforms. The definition is as before

$$x(t) = \frac{1}{2p\, j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds$$

Consider the sketch below, where the integration line runs up the centre ROC. Closing the curve with a giant semicircle to the left – where $s < 0$ – allows the integral along that semicircle to go to zero for positive time (consider the exponent of e^{st} in the integrand). Consequently, the integral is equal to the sum of residues at poles to the left of the ROC. We saw this in inversion of the one-sided transform back in Section 3. Similar arguments show that we can close the curve with a giant semicircle to the right for negative time. The residue argument also holds, although the fact that the integration proceeds in the opposite direction produces a negative sign.



We can summarize the inversion as follows:

$$x(t) = \begin{cases} \sum_{k \in \{LSP\}} R_k(t), & t \geq 0 \\ -\sum_{k \in \{RSP\}} R_k(t), & t < 0 \end{cases}$$

where *LSP* is the set of poles on the left side of the ROC and *RSP* is the set of poles on the right side of the ROC.

We'll do an example. Consider the transform

$$X_1(s) = \frac{5}{s^2 + s - 6} = \frac{5}{(s+3)(s-2)}$$

with region of convergence $-3 < s < 2$. We saw this one in Section 5.1. The sum of residues to the left of the ROC gives

$$x(t) = \left.\frac{5e^{st}}{s-2}\right|_{s=-3} = -e^{-3t}, \quad t \geq 0$$

and the negative of the sum of residues to the right of the ROC gives

$$x(t) = -\left.\frac{5e^{st}}{s+3}\right|_{s=2} = -e^{2t}, \quad t < 0$$

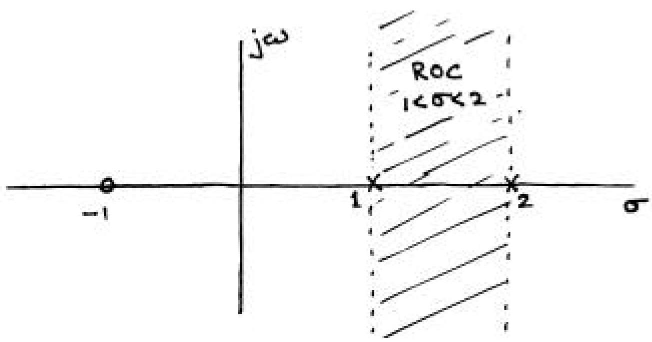
The result is correct, since it is consistent with the earlier example.

<u>Questions</u>

9. Calculate all three inverse transforms of $X(s) = \dfrac{s+1}{s^2 - 3s + 2}$

<u>Answers</u>

9. The easiest way to do this one is first to calculate both residues, then form the various combinations for the different regions of convergence. For reference, the pole-zero diagram and ROCs are illustrated below.



The residues are

$$R_1(t) = \left.\frac{(s+1)\,e^{st}}{s-2}\right|_{s=1} = -2e^{t} \quad \text{and} \quad R_2(t) = \left.\frac{(s+1)\,e^{st}}{s-1}\right|_{s=2} = 3e^{2t}$$

so we have for the three regions of convergence:

$$x_1(t) = \begin{cases} 0, & t \geq 0 \\ -R_1(t) - R_2(t) = 2e^{t} - 3e^{2t}, & t < 0 \end{cases}$$

$$x_2(t) = \begin{cases} R_1(t) = -2e^{t}, & t \geq 0 \\ -R_2(t) = -3e^{2t}, & t < 0 \end{cases}$$

$$x_3(t) = \begin{cases} R_1(t) + R_2(t) = -2e^{t} + 3e^{2t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$