3.5 Gaussian Quadratic Forms in Detection Problems

- The discussion so far has been limited to perfect CSI, because we couldn't analyze the effect of channel estimation error. Even for perfect CSI, we couldn't analyze the Rake detector in Section 3.2 unless the pulse autocorr'n function was zero for that delay.

- This section will:
  - cast common detection methods as calculation of quadratic forms
  - and develop an analytical tool to calculate the resulting BER

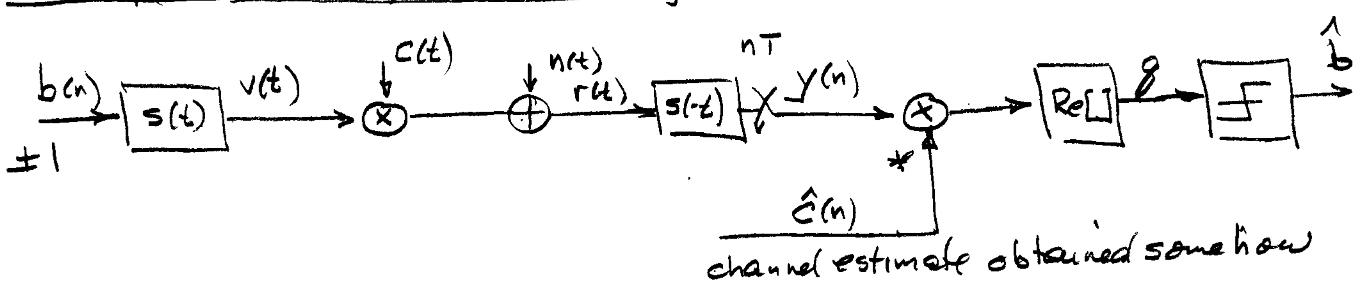
  The result is applicable to a wide range of situations.

- We apply the method in the next section to single user detection with imperfect CSI, and in later sections to multiuser detection

3.5.1 Central Role of Quadratic Forms

- Many detection schemes can be reduced to evaluation of the sign of a quadratic form in Gaussian variates. A few examples should be convincing, and there are several more in Appendix L.

- Example 1: PSK, flat fading

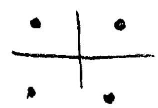


- Channel estimate is obtained from pilots, magic, etc
- Multiplication $g(n) = \mathrm{Re}[y(n)\,\hat{c}^*(n)]$ derotates the signal prior to slicing. It's a quadratic form:

$$g = [y^* \ \hat{c}^*]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y \\ \hat{c} \end{bmatrix}$$

error if $g<0$ when $b>0$

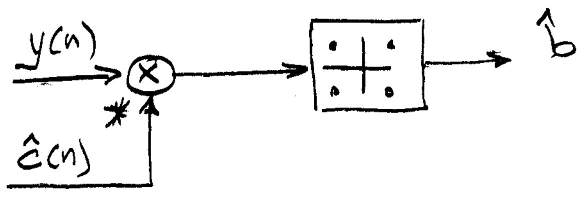
Hermitian matrix

- If QPSK  the above recovers the real bit

and $g_i(n) = \mathrm{Im}[y(n)\,\hat{c}^*(n)] = \mathrm{Re}[-j\,y(n)\,\hat{c}^*(n)]$ picks up the imag bit.

$$g_i = [y^* \ \hat{c}^*]\begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix}\begin{bmatrix} y \\ c \end{bmatrix}$$

Hermitian matrix



- Reality check: was that really the optimum way to use the channel estimate?

Suppose true gain and estimate are jointly Gaussian (zero mean, just for simplicity):

- corr'n coeff $\rho = \sigma_{c\hat{c}}^2 / \sigma_c \sigma_{\hat{c}}$
- so $c = \beta \hat{c} + e$, with $\beta = \rho \sigma_c / \sigma_{\hat{c}}$ (best linear est)

  and $\sigma_e^2 = \sigma_c^2 - |\beta|^2 \sigma_{\hat{c}}^2 = (1 - |\rho|^2)\sigma_c^2$
- MF output is conditionally Gaussian (conditioned on data bit, allows for multiamplitude):

  $$y = cAb + \nu = \beta \hat{c} A b + eAb + \nu$$

  with conditional mean and variance

  $$\mu_{y|\hat{c},b} = \beta \hat{c} A b \qquad \sigma_{y|\hat{c},b}^2 = \sigma_e^2 A^2 |b|^2 + N_0$$

The ML detector for b does

$$\arg\max_b \, p(y \mid \hat{c}, b)$$

$$= \arg\max_b \frac{1}{2\pi(\sigma_e^2 A^2 |b|^2 + N_0)} \exp\left(-\frac{1}{2} \frac{|y - \beta \hat{c} A b|^2}{(\sigma_e^2 A^2 |b|^2 + N_0)}\right)$$

log likelihood

$$\arg\max_b -\frac{1}{2} \frac{|y - \beta \hat{c} A b|^2}{(\sigma_e^2 A^2 |b|^2 + N_0)} - \ln\left(\sigma_e^2 A^2 |b|^2 + N_0\right) - \ln 2\pi$$

dropping hypothesis independent pieces ($|b|^2 = 1$ if PSK)

$$\arg\min_{b \in \{-1,1\}} |y - \beta \hat{c} A b|^2$$

$$= \arg\min_b \left(|y|^2 - 2\,\mathrm{Re}[y \hat{c}^* \beta^* b^*] A + |\beta \hat{c}|^2 A^2 |b|^2 \right.$$

$$= \arg\max_b \mathrm{Re}[y \hat{c}^* \beta^* b^*]$$

Usually β is positive, real (no fixed phase offset between c, $\hat{c}$. If b is real, then

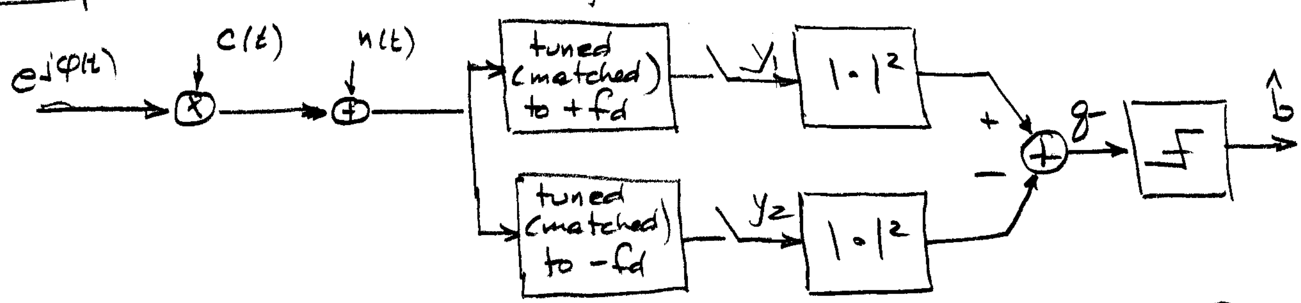
$$\hat{b} = \arg\max_b \mathrm{Re}[y\hat{c}^*]\, b = \mathrm{sgn}\left[\mathrm{Re}[y\hat{c}^*]\right]$$

error if $\mathrm{Re}[y\hat{c}^*] < 0$ when $b > 0$

So it was the right way to operate the detector.

- Summary: the optimum detector calculates a quadratic form g (or g_r and g_i) and makes an error if $g<0$ when $b>0$ (or $g>0$ when $b<0$).

- Example 2: FSK – flat fading

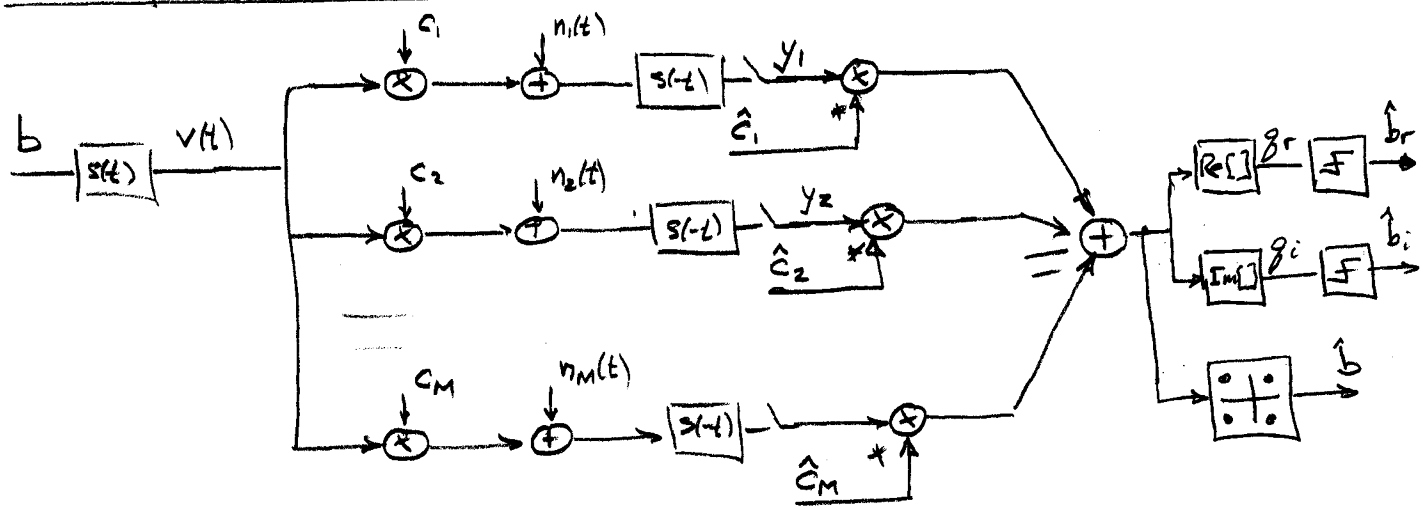


- The decision variable $g = |y_1|^2 - |y_2|^2$ is a quad form

$$\begin{bmatrix} y_1^* & y_2^* \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Variates y_1, y_2 are conditionally Gaussian. True for static channel, too; non zero mean plus Gaussian noise.
- Matrix is Hermitian.

• Example 3: Single user, diversity reception, BPSK



– It uses a quadratic form. Define $\underline{z} = \begin{bmatrix} y_1 \\ \hat{c}_1 \\ y_2 \\ \hat{c}_2 \\ \vdots \\ y_M \\ \hat{c}_M \end{bmatrix}$ so that

$$g = \frac{1}{2} \underline{z}^{+} Q \underline{z}$$

where

$$Q = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 1 & 0 \end{bmatrix}$$

Again Q is Hermitian, elements of $\underline{z}$ are conditionally jointly Gaussian, and an error if $g < 0$ when $b > 0$.

– But is this the best detector?

We have $c_m = \beta_m \hat{c}_m + e_m$ — for simplicity, $\beta_m = \beta$

$\sigma_e^2 = (1 - |\rho|^2)\sigma_c^2$

Then $\underline{c} = \beta \underline{\hat{c}} + \underline{e}$

$\mu_{\underline{c}|\underline{\hat{c}}} = \beta \underline{\hat{c}}$ $\quad R_{\underline{c}|\underline{\hat{c}}} = \sigma_e^2 I_M$

Also, $\underline{y} = \underline{c} A b + \underline{v} = \beta A b \hat{\underline{c}} + A b \underline{e} + \underline{v}$

so $\mu_{y|\hat{c},b} = \beta A b \hat{\underline{c}}$, $R_{y|\hat{c},b} = \left(A^2 |b|^2 \sigma_e^2 + N_0\right) I_M = \left(A^2 \sigma_e^2 + N_0\right) I_M$

- ML detector forms

$$\arg\max_b p(\underline{y}|\hat{\underline{c}}, b) = \frac{\exp\left(-\frac{1}{2}(\underline{y}-\mu_{y|\hat{c}b})^+ R_{y|\hat{c}b}^{-1} (\underline{y}-\mu_{y,\hat{c},b})\right)}{(2\pi)^M \, |R_{y|\hat{c}b}|}$$

$$= \arg\max_b \left[-\frac{1}{2}\left(\underline{y}^+ R_{y|\hat{c}b}^{-1} \underline{y} - 2\,\mathrm{Re}\left[\mu_{y|\hat{c}b}^+ R_{y|\hat{c}b}^{-1} \underline{y}\right] + \underbrace{\mu_{y|\hat{c}b}^+ R_{y|\hat{c}b}^{-1} \mu_{y|\hat{c}b}}_{|b|^2 = 1}\right)\right]$$

$$= \arg\max_b \mathrm{Re}\left[\mu_{y|\hat{c}b}^+ R_{y|\hat{c}b}^{-1} \underline{y}\right]$$

$$= \mathrm{sgn}\left[\mathrm{Re}\left[\beta^* \hat{\underline{c}}^+ \underline{y}\right]\right] = \mathrm{sgn}\left[\mathrm{Re}\left[\hat{\underline{c}}^+ \underline{y}\right]\right] \quad \text{for } \beta \text{ positive, real}$$

- Observations

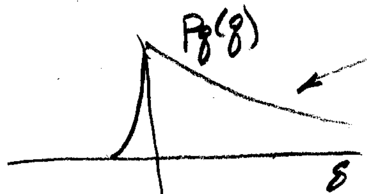
In these examples,

- the channel gain can be Gaussian, zeromean (Rayleigh) or nonzero mean (Rice), or it can be constant
- an error is made if a variable q, expressible as a Hermitian quadratic form, is negative when it should have been positive.
- the variates in the quadform are Gaussian when conditioned on the transmitted signal.
- if more than one data bit is involved ($\underline{b}$) then calculate the error prob for each choice of interfering bits, and average them.

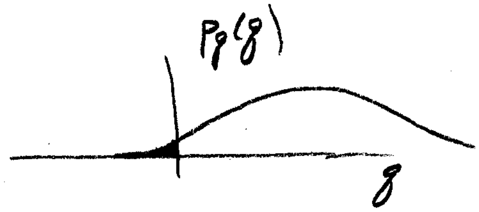
– In these quadratic forms, Q is Hermitian, so g is real. If

Q is	pos def	then	$g > 0$
	pos semidef		$g \geq 0$
	neg def		$g < 0$
	neg semidef		$g \leq 0$
	indefinite		$-\infty < g < \infty$

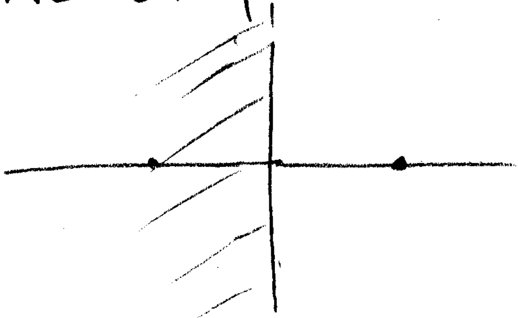
• We are interested in the pdf of the decision variable g:

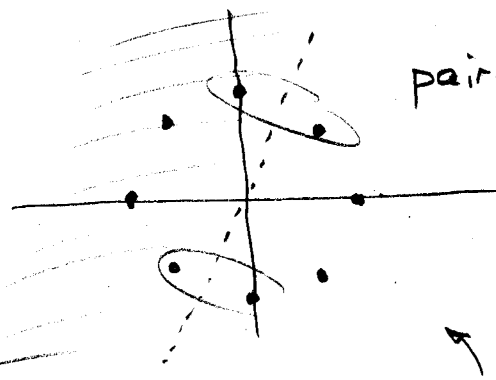


typical for indefinite Q



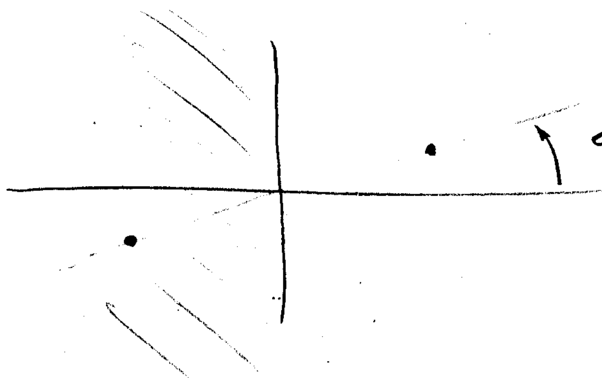
and, in particular, the prob that $g < 0$ since we set our problem up so that $d < 0$ corresponds to an error.





pairwise

Appendix L



angular displacement through frequency offset

3.5.2 Characteristic Function of Quadratic Form

- Read [Schw66, App B], reproduced in Appendix M and summarized below.

- For a Hermitian quadratic form in complex Gaussian random variables)

$$q = \tfrac{1}{2}\, \underline{x}^{+} Q\, \underline{x}, \qquad E[\underline{x}] = \underline{\mu} \qquad R = \tfrac{1}{2} E[\underline{x}\,\underline{x}^{+}]$$

define the characteristic function as the 2-sided Laplace transform of its pdf (Appendix N)

$$M_q(s) = \int_{-\infty}^{\infty} p_q(q)\, e^{-sq}\, dq$$

Then

$$M_q(s) = \frac{\exp\left(-\tfrac{1}{2} s\, \underline{\mu}^{+}(Q^{-1} + sR)^{-1} \underline{\mu}\right)}{\det(I + sRQ)}$$

Notation:

[Schw66]	here
F	$\tfrac{1}{2}Q$
R^*	R

and for Rayleigh

$$M_q(s) = \frac{1}{\det(I + sRQ)}$$

$$= \frac{1}{\text{poly}(s)}$$

with real roots; obviously related to the characteristic poly

- Transformation makes it simpler still:

$$\Phi = [\phi_1 | \phi_2 \cdots | \phi_N] \quad \text{(unitary) matrix of eivecs of } RQ)$$

then

$$\frac{1}{\det(I + sRQ)} = \frac{1}{|\Phi| \cdot |I + sB| \cdot |\Phi^{-1}|} = \frac{1}{|I + sB|}$$

where $B = \mathrm{diag}(\beta_1, \beta_2, \cdots \beta_N)$ are eivals of RQ.

So
$$M_g(s) = \frac{1}{\prod_{i=1}^{N} (1 + s\beta_i)} = \prod_{i=1}^{N} \frac{-\lambda_i}{s - \lambda_i}$$

where $\lambda_i = -1/\beta_i$ are the eivals of $-(RQ)^{-1} = -Q^{-1}R^{-1}$

so the characteristic function is a rational polynomial with real roots, and the pdf is a sum of exponentials.

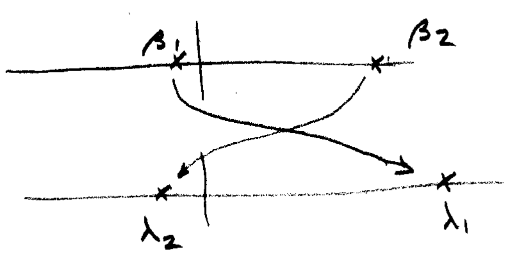
- Example $\boxed{MF} \xrightarrow{y} \otimes \longrightarrow \boxed{Re[\;]} \xrightarrow{g}$, with $\hat{\varepsilon}^*$ into $\otimes$ $\qquad \alpha$ corr'n coeff

$$R = \begin{bmatrix} \sigma_y^2 & \sigma_{y\hat{\varepsilon}}^2 \\ \sigma_{y\hat{\varepsilon}}^{2*} & \sigma_{\hat{\varepsilon}}^2 \end{bmatrix} = \begin{bmatrix} \sigma_y^2 & \alpha \sigma_y \sigma_{\hat{\varepsilon}} \\ \alpha^* \sigma_y \sigma_{\hat{\varepsilon}} & \sigma_{\hat{\varepsilon}}^2 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$RQ = \begin{bmatrix} \alpha \sigma_y \sigma_{\hat{\varepsilon}} & \sigma_y^2 \\ \sigma_{\hat{\varepsilon}}^2 & \alpha^* \sigma_y \sigma_{\hat{\varepsilon}} \end{bmatrix}$$ and now get the eigenvals

$$|RQ - \beta I| = \begin{vmatrix} \alpha \sigma_y \sigma_{\hat{\varepsilon}} - \beta & \sigma_y^2 \\ \sigma_{\hat{\varepsilon}}^2 & \alpha^* \sigma_y \sigma_{\hat{\varepsilon}} - \beta \end{vmatrix} = 0$$

$$\beta = \sigma_y \sigma_{\hat{\varepsilon}} \left(\alpha_r \pm \sqrt{1 - \alpha_i^2} \right)$$ eivals real, pos & neg pair

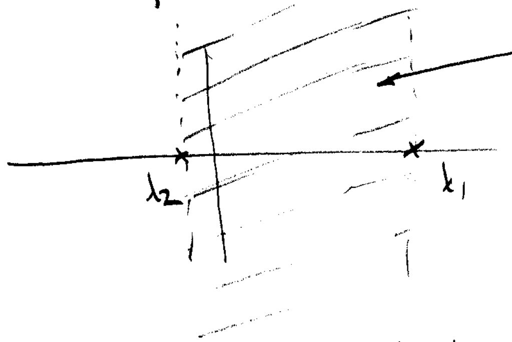


$$\beta = \sigma_y \sigma_{\hat{\varepsilon}} (\alpha \mp 1)$$
for α real

- Inversion of $M_g(s) = \prod_{i=1}^{N} \frac{-\lambda_i}{s-\lambda_i}$ can be accomplished by standard methods (residues, partial fractions). By residues, it is

$$p_g(g) = \begin{cases} -\sum\limits_{\lambda_i \in RP} \operatorname{Res}\left[M_g(s)e^{sg}\right]_{\lambda_i}, & g \leq 0 \\[2ex] \sum\limits_{\lambda_i \in LP,0} \operatorname{Res}\left[M_g(s)e^{sg}\right]_{\lambda_i}, & g > 0 \end{cases}$$

- Example (continued from p. 3.5.9)



region of convergence (Appendix N)

$$\lambda_1 = \frac{1}{\sigma_y \sigma_{\hat{\varepsilon}} \left(\sqrt{1-\alpha_i^2} - \alpha_r\right)}$$

$$\lambda_2 = \frac{-1}{\sigma_y \sigma_{\hat{\varepsilon}} \left(\sqrt{1-\alpha_i^2} + \alpha_r\right)}$$

$$M_g(s) = \frac{\lambda_1 \lambda_2}{(s-\lambda_1)(s-\lambda_2)}$$

Inversion by residues:

- for $g \leq 0$, $-\operatorname{Res}\left[M_g(s)e^{sg}\right]_{\lambda_1} = \frac{-\lambda_1\lambda_2}{\lambda_1-\lambda_2} e^{\lambda_1 g}$

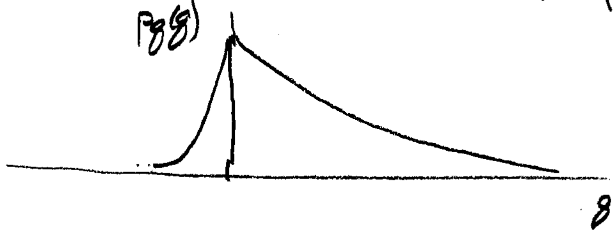
$$= \frac{1}{2\sigma_y \sigma_{\hat{\varepsilon}} \sqrt{1-\alpha_i^2}} \exp\left(\frac{g}{\sigma_y \sigma_{\hat{\varepsilon}} \left(\sqrt{1-\alpha_i^2} - \alpha_r\right)}\right)$$

- for $g > 0$, $\operatorname{Res}\left[M_g(s)e^{sg}\right]_{\lambda_2} = \frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1} e^{\lambda_2 g}$

$$= \frac{1}{2\sigma_y \sigma_{\hat{\varepsilon}} \sqrt{1-\alpha_i^2}} \exp\left(\frac{-g}{\sigma_y \sigma_{\hat{\varepsilon}} \left(\sqrt{1-\alpha_i^2} + \alpha_r\right)}\right)$$

– together,

$$p_g(g) = \frac{1}{2\sigma_y\sigma_\varepsilon\sqrt{1-\alpha_i^2}} \begin{cases} \exp\left(\frac{g}{\sigma_y\sigma_\varepsilon\left(\sqrt{1-\alpha_i^2}-\alpha_r\right)}\right), & g \le 0 \\ \exp\left(\frac{-g}{\sigma_y\sigma_\varepsilon\left(\sqrt{1-\alpha_i^2}+\alpha_r\right)}\right), & g > 0 \end{cases}$$



- Probability $g \le 0$ can be obtained by integration as

$$\int_{-\infty}^{0} p_g(g)\,dg = \frac{1}{2}\,\frac{1-|\alpha|^2}{1+\alpha_r\sqrt{1-\alpha_i^2}} \longrightarrow \frac{1}{2}(1-\alpha) \text{ for } \alpha \text{ pos, real}$$

<u>or</u> directly from the characteristic function

$$\Pr[g<0] = -\sum_{\lambda_i \in RP} \operatorname{Res}\left[M_g(s)/s\right]_{\lambda_i}$$

$$= \left[\frac{-\lambda_1\lambda_2}{s(s-\lambda_2)}\right]_{\lambda_1} = \frac{1}{2}\,\frac{1-|\alpha|^2}{1+\alpha_r\sqrt{1-\alpha_i^2}} \longrightarrow \frac{1}{2}(1-\alpha)$$

- If there's a bias δ, $\Pr[g<\delta] = -\sum_{\lambda_i \in RP} \operatorname{Res}\left[M_g(s)e^{s\delta}/s\right]_{\lambda_i}$

- For multiple poles, differentiation in residues or partial fractions is cumbersome. Numerical integration (Appendix N) becomes numerically unstable. See appendix of [Welb01] for procedure than avoids drudgery and numerical instability

3.5.3 A Special Solution

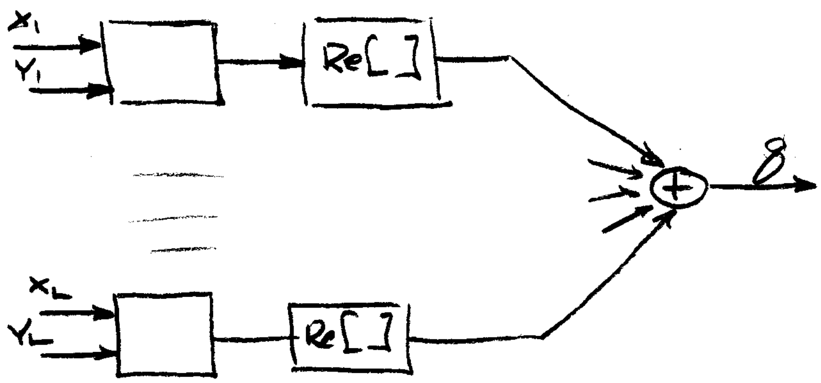
- Read J Proakis Digital Communications App B

- For the important case where Q and R are block diagonal, with constant 2×2 block

$$Q = \begin{bmatrix} Q' & & & 0 \\ & Q' & & \\ & & Q' & \\ 0 & & & Q' \end{bmatrix} \qquad R = \begin{bmatrix} R' & & & 0 \\ & R' & & \\ & & R' & \\ 0 & & & R' \end{bmatrix}$$

2×2 $\qquad$ 2×2 $\qquad$ L blocks

it corresponds to identical and separate processing of L identically distributed, statistically independent channels



Proakis has provided a complete solution for both Rayleigh and Rice cases: $\Pr[g<0]$

Even in the straightforward Rayleigh case, it is useful, since the poles now have multiplicity L, and calculation of residues is a nuisance:

$$M_g(s) = \frac{1}{\det(I + sRQ)} = \frac{1}{|I' + sR'Q'|^L}$$

$$= \prod_{i=1}^{2} \left(\frac{-\lambda_i}{s-\lambda_i}\right)^L$$

- Formulation:

$$g = A|X|^2 + B|Y|^2 + CXY^* + C^*X^*Y \qquad \text{with } A, B \text{ real}$$

$$= \frac{1}{2}\underline{\xi}^{+} Q' \underline{\xi} \quad \text{where } \underline{\xi} = \begin{bmatrix} X \\ Y \end{bmatrix}, \quad Q' = 2\begin{bmatrix} A & C^* \\ C & B \end{bmatrix}$$

The statistics of X, Y are Gaussian with

$$\overline{\underline{\xi}} = \begin{bmatrix} \bar{X} \\ \bar{Y} \end{bmatrix}$$

and

$$\overline{(\underline{\xi} - \overline{\underline{\xi}})(\underline{\xi} - \overline{\underline{\xi}})^{+}} = E\begin{bmatrix} |X-\bar{X}|^2 & (X-\bar{X})(Y-\bar{Y})^* \\ (X-\bar{X})^*(Y-\bar{Y}) & |Y-\bar{Y}|^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mu_{xx} & \mu_{xy} \\ \mu_{xy}^* & \mu_{yy} \end{bmatrix} = 2R$$

where $\mu_{xx} = 2\sigma_x^2 \quad \mu_{yy} = 2\sigma_y^2 \quad \mu_{xy} = 2\sigma_{xy}^2$

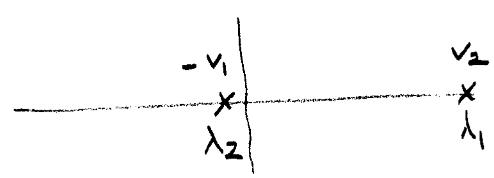
For diversity channels:

$$g = \sum_{k=1}^{L} g_k = \frac{1}{2}\sum_{k=1}^{L} \underline{\xi}_k^{+} Q' \underline{\xi}_k$$

- Then $P[g<0]$ is given by (B-21) in [Proa 95].

Not reproduced here, but note

Proakis		here
$-v_1$	$\longleftrightarrow$	$\lambda_2 < 0$
v_2	$\longleftrightarrow$	$\lambda_1 > 0$



error

$$w = \frac{A\mu_{xx} + B\mu_{yy} + C\mu_{xy} + C^*\mu_{xy}^*}{2(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}$$

poles are eigenvals of $-(RQ)^{-1}$

Note use of Marcum Q-function, the area under the tail of a Rice pdf with $\sigma = 1$, $K = a^2/2$:

$$Q(a,b) = \int_b^\infty x\, e^{-(x^2+a^2)/2}\, I_0(ax)\, dx$$

$$Q(0,b) = \int_b^\infty x\, e^{-x^2/2}\, dx = e^{-b^2/2}$$

- Simplification of Proakis result if Rayleigh (zero mean) but multichannel:

$$a = b = 0 \qquad I_n(0) = \delta_{n0} \qquad Q(0,0) = 1$$

$$\Pr[g<0] = \frac{1}{1+(v_2/v_1)^{2L-1}} \sum_{k=0}^{L-1} \binom{2L-1}{k} \left(\frac{v_2}{v_1}\right)^k$$

where v_2/v_1 is the positive ratio of the larger pole to the negative of the smaller one. $v_2/v_1 = -\lambda_1/\lambda_2 > 0$

Example: on pp 3.5.9 – 3.5.11, we used $A = B = 0$, $C = 1/2$

$$\frac{v_2}{v_1} = \frac{\sqrt{1-\alpha_i^2} + \alpha_r}{\sqrt{1-\alpha_i^2} - \alpha_r}$$

- Simplification if single channel ($L=1$) but Rice (non zero mean).

$$\Pr[g<0] = Q(a,b) - \frac{v_2/v_1}{1+v_2/v_1}\, I_0(ab)\, e^{-(a^2+b^2)/2}$$

where a, b defined in Proakis

An alternative for single channel Rice, adapted from [Schw 66]

$$\alpha = \frac{\sigma_{xY}^2}{\sigma_x \sigma_Y} = \alpha_r + j\alpha_i \quad \text{correlation coefficient}$$

$$\begin{Bmatrix} a^2 \\ b^2 \end{Bmatrix} = \frac{|\bar{x}|^2/\sigma_x^2 + |\bar{Y}|^2/\sigma_Y^2 - 2\alpha_i \, \mathrm{Im}[\bar{x}\bar{Y}^*]/\sigma_x\sigma_Y \mp 2\sqrt{1+\alpha_i^2}\, \mathrm{Re}[\bar{x}\bar{Y}^*]/\sigma_x\sigma_Y}{4(1-\alpha_i^2)}$$

$$\Pr[g<0] = Q(a,b) - \frac{\sqrt{1-\alpha_i^2}+\alpha_r}{2\sqrt{1-\alpha_i^2}} I_0(ab)\, e^{-(a^2+b^2)/2}$$

- Simplification for single channel Rayleigh

$$\Pr[g<0] = \frac{1}{1+ V_2/V_1}$$

Example: Again $A=B=0$, $C=\frac{1}{2}$

$$\Pr[g<0] = \frac{1}{2}\left(1 - \frac{\alpha_r}{\sqrt{1-\alpha_i^2}}\right)$$

$$= \frac{1}{2}(1-\alpha) \quad \text{if } \alpha \text{ is real, positive,}$$

as we saw in Section 3.5.2