4.2 Rank and Dimensionality

- We have seen in Section 3.7 that ML users can be supported when they all have the same pulse shape (signature sequence). What does assignment of of different signatures to users buy us?
  - If those signatures were orthogonal to each other and the time translates, then we'd have a proportional increase in capacity. Too bad they are not.
- In this section, we'll see that
  - the precombining method (multiuser separation before Rake/MRC) is vulnerable to linear dependences among the sequences and their translates
  - but the postcombining approach (classical Rake/MRC before MUD) is much more robust — the path gains help.

- First, precombining (MUD before Rake/MRC). We have

$$\underline{y} = \underline{R}\, C A \underline{b} + \underline{v}$$

and we form $\underline{z} = \underline{R}^{-1}\underline{y} = C A \underline{b} + \underline{v}$

– This requires that $\underline{R}$ be nonsingular. Is it?

structure:

$$\underline{R} = \begin{bmatrix} R & & 0 \\ & R & \\ 0 & & \ddots \quad R \end{bmatrix} \qquad \text{so } \underline{R}^{-1} = \text{diag}(R^{-1}, R^{-1}, \cdots R^{-1})$$

and structure of R:

$$R = \begin{bmatrix} R(0,0) & R(0,1) & \cdots & 0 \\ R(1,0) & R(1,1) & \ddots & \\ & \ddots & \ddots & \\ 0 & & & R(N-1,N-1) \end{bmatrix}$$

N blocks × N blocks

– For simplicity, assume short codes and ignore the block super and sub diagonals, so that

$$R = \begin{bmatrix} R & & 0 \\ & R & \\ 0 & & \ddots \quad R \end{bmatrix}$$

Nonsingular R requires nonsingular R, so focus on structure and rank of R

– Signal correlation matrix $R = S^{\dagger}S$, where

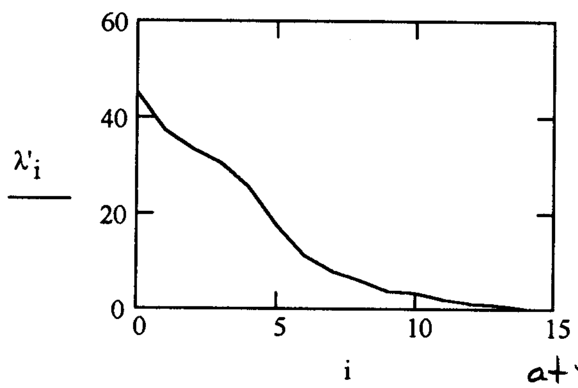
$$S = \underbrace{\begin{bmatrix} \underline{s}_1 & \underline{s}_1 & \underline{s}_1 & \underline{s}_2 & \underline{s}_2 & \underline{s}_2 & \cdots & \underline{s}_K & \underline{s}_K & \underline{s}_K \end{bmatrix}}_{KL}$$

$N_s N_c$ + allowance for delay spread and asynchronous.

The number of LI columns is limited to the number of rows — conservatively, $N_s N_c$; even more conservatively, $2N_c$, since $N_s > 2$ rapidly increases risk of dependence.

– Looks as though KL is the load on this matrix, and that $KL \lesssim 2N_c$ for nonsingular R.

$N_s = 1$ $\tau_k = 0, \forall k$ Plots of eigenvalues of R.



$N_c = 15$

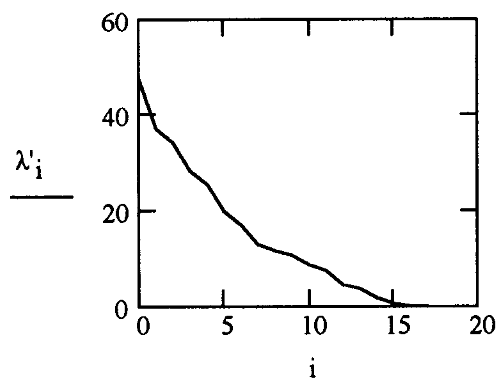
$K = 5$

$L = 3$

$N = 1$

eigenvalratio(R) = 401.289

at the edge of working



$N_c = 15$

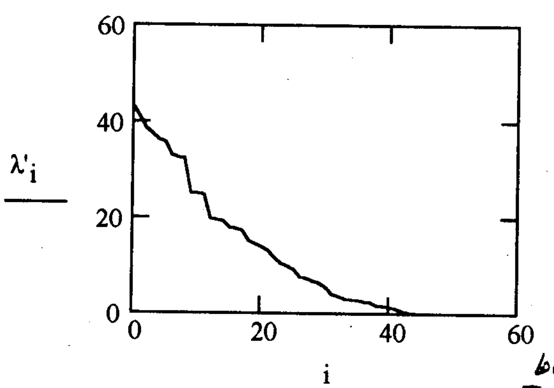
$K = 6$

$L = 3$

$N = 1$

eigenvalratio(R) = $8.732 \cdot 10^{15}$

just one more user



$N_c = 15$

$K = 5$

$L = 3$

$N = 3$

eigenvalratio(R) = 367.714

back to 5 users, but 3 symbol times

Predetection limits users to

$\boxed{K \approx N_c / L}$ for $N_s = 1$

Rank of R does not exceed $\approx N_c N$

- Next, classical post combining. We form

$$\underline{\zeta} = C^{+} \underline{R}\, C A \underline{b} + \eta$$

and (for decorrelation, at least) require $C^{+}\underline{R}C$ to be non-sing. Is it?

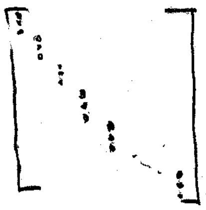
– We have $\underline{R} = \begin{bmatrix} R & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix}$ $MNKL \times MNKL$

with rank (from previous discussion) not exceeding

$$\text{rank}(\underline{R}) \leq MNN_c \quad (\text{for } N_s = 1 \text{ sample/chip})$$

– We also have

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_M \end{bmatrix}$$



$MNKL \times NK$ $NKL \times NK$

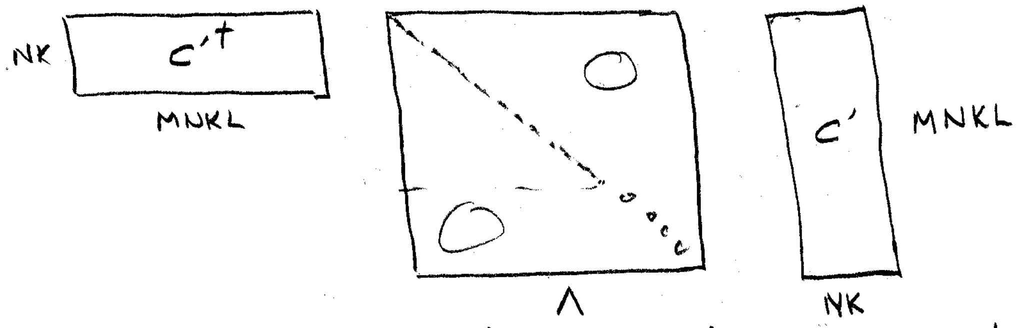
so it has full column rank.

– Ideally, all those sequences and translates are orthogonal, so $\underline{R} = I_{MNKL}$, which would make $C^{+}\underline{R}C = C^{+}C$, for perfect Rake/MRC. More realistically, some degeneracy in $\underline{R}$, so examine in detail.

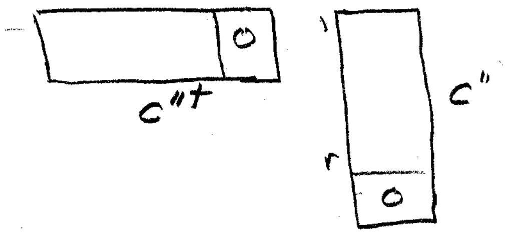
- Eigendecomposition of $\underline{R} = \Phi \Lambda \Phi^\dagger$, where Φ is unitary matrix of eigenvectors, Λ diag eigenvals

$$\underline{c}^\dagger \underline{R}\,\underline{c} = \underline{c}^\dagger \Phi \Lambda \underbrace{\Phi^\dagger \underline{c}}_{\underline{c}'} = \underline{c}'^\dagger \Lambda\, \underline{c}'$$

c' has full column rank, inherited from c



The effect of zero eigenvalues is to remove bottom rows of c', leaving r (the rank of $\underline{R}$) non zero entries.



$c'' = \Lambda^{\frac{1}{2}} c$

$r \approx MNN_c$

With probability one, c'' retains full column rank if

$$r \geq NK.$$

and this keeps $c^\dagger \underline{R} c$ nonsingular, as required.

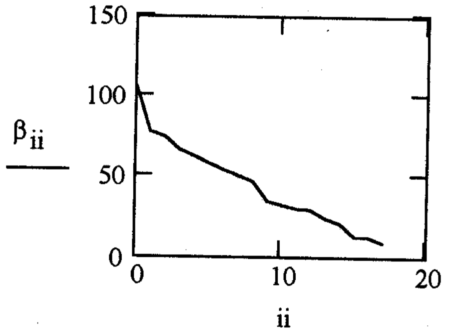
- Therefore maximum K is determined by $r = NK$

$$MNN_c \approx NK$$

or $\boxed{K \lesssim M N_c \text{ users}}$ for $N_c = 1$

Much better than precombining!

- The multiple measurements on paths and antennas preserves and extends dimensionality.
  - If $\underline{R}$ has full rank and all eigenvals about the same, then full diversity effect $(M \times L)$
  - As eigenvalues of $\underline{R}$ become profiled, some small, we lose some of the diversity effect
  - When the rank of $\underline{R}$ is down to NK, then each bit gets single diversity — but it's still hanging on!
- Typical results for $N_s = 1$ sample/chip, synchronous users, single antenna.



$N_c = 15$

$K = 6$

$L = 3$

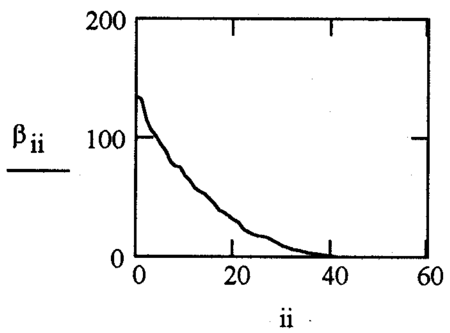
$N = 3$

$M = 1$

Easy inversion of $C^+ \underline{R} C$ with this eigenval ratio, even though R^{-1} does not exist (so precombining fails).

$$\text{eigenvalratio}\left(\overline{\left(C^T\right)} \cdot R \cdot C\right) = 11.903$$

$$\text{eigenvalratio}(R) = -2.048 \cdot 10^{15} \quad \text{rank}(R) = 47$$



$N_c = 15$

$K = 14$

$L = 3$

$N = 3$

$M = 1$

Tougher inversion of $C^+ RC$ and lots of noise enhancement, but K is almost equal to N_c.

A second antenna will double the number of users or reduce the noise enhancement.

$$\text{eigenvalratio}\left(\overline{\left(C^T\right)} \cdot R \cdot C\right) = 723.385$$

$$\text{eigenvalratio}(R) = -2.271 \cdot 10^{15} \quad \text{rank}(R) = 47$$