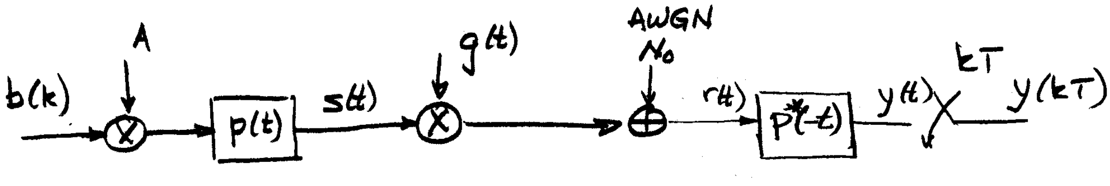
UNIVERSITY OF CANTERBURY
Dept. of Electrical and Computer Engineering

ENEL 673

Solutions to Assignment 2 **April, 2002**

1. SNR Calculations



(a) We know that complex envelope power and bandpass power are related by a factor of 1/2, so the energies over one symbol duration *T* have the same relationship. This gives us

$$E_b = \frac{1}{2}\cdot E\left[\int_0^T \left(|s(t)|\right)^2 dt\right]$$

where I have taken the interval [0,*T*] as representative, and the expectation is over the data ensemble. Substituting for the complex envelope, we have

$$E_b = \frac{1}{2}\cdot A^2\cdot E\left[\int_0^T \left(\left|\sum_k b(k)\cdot p(t-k\cdot T)\right|\right)^2 dt\right]$$

$$= \frac{1}{2}\cdot A^2\cdot E\left(\int_0^T \sum_k\sum_i b(k)\cdot\overline{b(i)}\cdot p(t-k\cdot T)\cdot\overline{p(t-i\cdot T)}\,dt\right)$$

Bring the expectation inside the integral and summations, then use the fact that the data symbols are independent and have unit amplitude, so $E\left(b(k)\cdot\overline{b(i)}\right) = \delta_{i,k}$. The result is

$$E_b = \frac{1}{2}\cdot A^2\cdot\int_0^T \sum_k \left(|p(t-k\cdot T)|\right)^2 dt = \frac{1}{2}\cdot A^2\cdot\sum_k\int_0^T \left(|p(t-k\cdot T)|\right)^2 dt = \frac{1}{2}\cdot A^2$$

From this, $A = \sqrt{2\cdot E_b}$

(b) Consider the noise component at the output of the matched filter, before sampling. The situation is just complex white noise into a linear filter, so all we need is the autocorrelation function of the filter output. The filter can be described by

$$h(t) = \overline{p(-t)} \qquad H(f) = \overline{P(f)}$$

The power spectrum at the output is $S_y(f) = N_o \cdot (|H(f)|)^2 = N_o \cdot (|P(f)|)^2$

which makes the autocorrelation function

$$R_y(\tau) = N_o \cdot R_p(\tau) \quad \text{where} \quad R_p(\tau) = \int_{-\infty}^{\infty} p(t) \cdot \overline{p(t-\tau)}\, dt$$

We could equally well work in the time domain, using . Then

$$R_y(\tau) = E\left(y(t) \cdot \overline{y(t-\tau)}\right) = E\left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) \cdot n(t-\alpha) \cdot \overline{h(\beta)} \cdot \overline{n(t-\tau-\beta)}\, d\alpha\, d\beta\right)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha) \cdot \overline{h(\beta)} \cdot N_o \cdot \delta(\tau + \beta - \alpha)\, d\alpha\, d\beta = N_o \cdot \int_{-\infty}^{\infty} h(\alpha) \cdot \overline{h(\alpha - \tau)}\, d\alpha$$

$$= N_o \cdot R_p(\tau)$$

Now, back to the original question. The variance of the noise in the matched filter output is

$$\sigma_v^2 = R_y(0) = N_o \cdot R_p(0) = N_o$$ since the pulse has unit energy

and the autocorrelation function of the samples (which I'll denote with lower case *r* to distinguish it from continuous time) is

$$r_y(k) = R_y(k \cdot T) = N_o \cdot R_p(k \cdot T)$$

A common special case: Nyquist pulses, samples spaced by the symbol time *T*. Then

$$R_p(k \cdot T) = \delta_{k,0} \quad \text{and} \quad r_y(k) = N_o \cdot \delta_{k,0}$$

(c) If the channel gain c=1, then the signal component at the matched filter output is

$$y(t) = \int_{-\infty}^{\infty} s(\alpha) \cdot h(t-\alpha)\, d\alpha = \int_{-\infty}^{\infty} s(\alpha) \cdot \overline{p(\alpha - t)}\, d\alpha = A \cdot \sum_i b(i) \cdot R_p(t - i \cdot T)$$

and the sample at t=kT is

$$y(k\cdot T) = A\cdot\sum_i b(i)\cdot R_p[(k-i)\cdot T]$$ and if Nyquist pulses $$y(k\cdot T) = A\cdot b(k)$$

The variance of the signal component is $$\frac{1}{2}\cdot E\left[\left(|y(k\cdot T)|\right)^2\right] = \frac{A^2}{2} = E_b$$

Finally, the ratio of signal variance to noise variance (signal power to noise power is)

$$\Gamma = \frac{A^2}{2}\cdot\frac{1}{N_o} = \frac{E_b}{N_o}$$ and for constellations higher than binary $$\Gamma = \frac{E_s}{N_o}$$

(d) From part (c) and the expression for the received signal, we have

$$y(k\cdot T) = \int_{-\infty}^{\infty} r(\alpha)\cdot\overline{p(\alpha-k\cdot T)}\,d\alpha = A\cdot\sum_i b(i)\cdot\int_{-\infty}^{\infty} g(\alpha)\cdot p(\alpha)\cdot\overline{p(\alpha-k\cdot T)}\,d\alpha + \nu(k\cdot T)$$

Examine the integrand. If $g(\alpha)$ varies slowly compared with the other two factors, then we can approximate the integral as

$$g(k\cdot T)\cdot\int_{-\infty}^{\infty} p(\alpha)\cdot\overline{p(\alpha-k\cdot T)}\,d\alpha = g(k\cdot T)\cdot R_p(k\cdot T)$$

This gives

$$y(k\cdot T) = A\cdot g(k\cdot T)\cdot\sum_i b(i)\cdot R_p[(k-i)\cdot T] + \nu(k\cdot T)$$

and with Nyquist pulses, $$y(k\cdot T) = A\cdot g(k\cdot T)\cdot b(k) + \nu(k\cdot T)$$

(e) If the gain is random, we return to the question of signal variance, this time with r(t), instead of y(t). We have the variance of the signal component as

$$\frac{1}{2}\cdot E\left[\left(|g(t)\cdot s(t)|\right)^2\right] = \sigma_s^{\,2}\cdot E\left[\left(|g(t)|\right)^2\right]$$

Therefore, to keep the signal power unchanged, we need $$\frac{1}{2}\cdot E\left[\left(|g(t)|\right)^2\right] = \frac{1}{2}$$

If *g* is zero mean, this quantity is the variance, and $$\sigma_g^{\,2} = \frac{1}{2}$$

This is an awkward sort of result. It illustrates the fact that the factor of 1/2 applies to both signals ($s(t)$ and $g(t)$) together, not individually, when we calculate power at the receiver. You get used to it after a while, but it is always a nuisance.

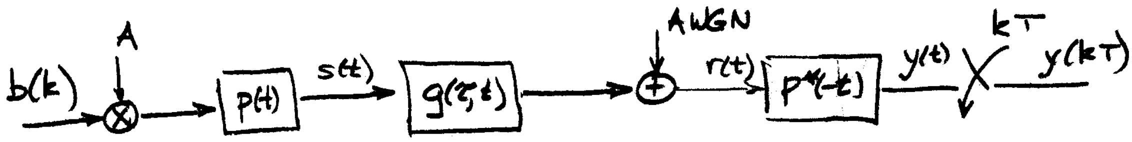
(f) If the conditions of (d) and (e) hold, then the signal to noise ratio in the MF output samples is

signal: $\frac{1}{2}\cdot E\left[\left(|y(k\cdot T)|\right)^2\right] = \frac{1}{2}\cdot E\left[\left(|A\cdot g(k\cdot T)\cdot b(k)|\right)^2\right] = A^2\cdot \sigma_g^{\;2} = E_s$

noise: N_o

ratio: $\Gamma = \frac{E_s}{N_o}$

(g) The frequency-selective fading link is shown below.



We have the received signal as

$$r(t) = \sum_{i=0}^{L-1} g_i(t)\cdot s\left(t-\tau_i\right)$$

Its expected instantaneous power is

$$P_r(t) = \frac{1}{2}\cdot E\left[\left(|r(t)|\right)^2\right] = \frac{1}{2}\cdot E\left(\sum_{i=0}^{L-1}\sum_{k=0}^{L-1} g_i(t)\cdot \overline{g_k(t)}\cdot s\left(t-\tau_i\right)\cdot \overline{s\left(t-\tau_k\right)}\right)$$

where expectation is across the channel and data ensembles. Because the path gains are uncorrelated and zero-mean, we have

$$P_r(t) = \sum_{i=0}^{L-1} \sigma_i^{\;2}\cdot E\left[\left(\left|s\left(t-\tau_i\right)\right|\right)^2\right]$$ where $\sigma_i^{\;2}$ is the variance of $g_i(t)$

$$= \sum_{i=0}^{L-1} \sigma_i^{\;2}\cdot\left(A^2\cdot\sum_j\sum_k E\left(b(j)\cdot\overline{b(k)}\right)\cdot p\left(t-j\cdot T-\tau_i\right)\overline{p\left(t-k\cdot T-\tau_i\right)}\right)$$

$$= A^2\cdot\sum_{i=0}^{L-1}\sum_j \sigma_i^{\;2}\cdot\left(\left|p\left(t-j\cdot T-\tau_i\right)\right|\right)^2$$

which is periodic in t. Since the pulses have unit energy, the received energy across a symbol time is

$$E_s = \int_0^T P_r(t)\,dt = A^2 \cdot \sum_{i=0}^{L-1} \sigma_i^2 \cdot \int_0^T \sum_j \left(\left| p\left(t - j \cdot T - \tau_i\right) \right| \right)^2 dT = A^2 \cdot \sum_{i=0}^{L-1} \sigma_i^2$$

or

$$E_s = 2 \cdot E_b \cdot \sum_{i=0}^{L-1} \sigma_i^2$$

which gives the average received power as

$$P_{r\_av} = \frac{E_s}{T} = \frac{A^2}{T} \cdot \sum_{i=0}^{L-1} \sigma_i^2 = \frac{2 \cdot E_s}{T} \cdot \sum_{i=0}^{L-1} \sigma_i^2$$

If we want the transmitted and received powers, or energies per symbol, to be the same, then set

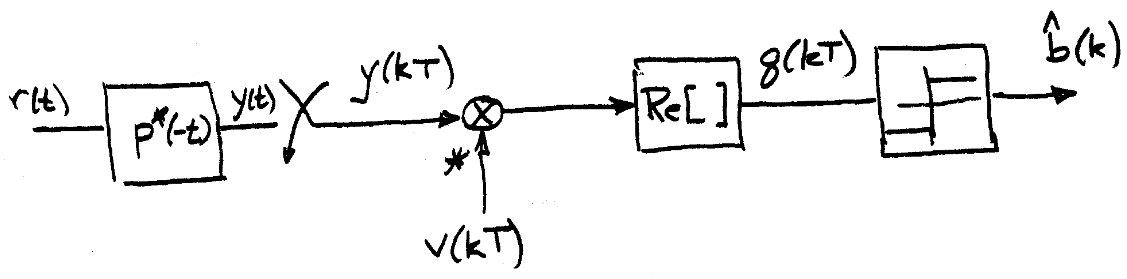
$$\sum_{i=0}^{L-1} \sigma_i^2 = \frac{1}{2}$$

To generalise a little, if the channel has a continuous impulse response, instead of discrete arrivals, then

$$\int_0^{\tau_{max}} P_g(\tau)\,d\tau = \frac{1}{2}$$

where $P_g(\tau)$ is the power delay profile (the continuous version of σ_i^2 vs. τ_i).

2. Coherent and Incoherent Detection



In this system, we have $y(k \cdot T) = \sqrt{2 \cdot E_b} \cdot g(k \cdot T) \cdot b(k) + n(k \cdot T)$

and $v(k \cdot T) = g(k \cdot T) + e(k \cdot T)$

with $\sigma_g^{\ 2} = \frac{1}{2}$ $\quad \sigma_n^{\ 2} = N_o$ and $\sigma_v^{\ 2} = \sigma_g^{\ 2} + \sigma_e^{\ 2}$

$$\sigma_y^{\ 2} = \frac{1}{2}\cdot E\left[\left(\left|y(k\cdot T)\right|\right)^2\right] = 2\cdot E_b\cdot \sigma_g^{\ 2} + N_o = E_b + N_o$$

(a) Correlation coefficients are the key to detector performance. Start with

$$\rho = \frac{\sigma_{gv}^{\ 2}}{\sigma_g\cdot \sigma_v}$$

First, the numerator: $\sigma_{gv}^{\ 2} = \frac{1}{2}\cdot g(k\cdot T)\cdot \overline{(g(k\cdot T) + e(k\cdot T))} = \sigma_g^{\ 2}$

From this,

$$\rho = \frac{\sigma_{gv}^{\ 2}}{\sigma_g\cdot \sigma_v} = \frac{\sigma_g^{\ 2}}{\sqrt{\sigma_g^{\ 2}\cdot\left(\sigma_g^{\ 2} + \sigma_e^{\ 2}\right)}} = \frac{1}{\sqrt{1 + \frac{\sigma_e^{\ 2}}{\sigma_g^{\ 2}}}} = \frac{1}{\sqrt{1 + \Gamma_r^{\ -1}}}$$

where we denote the reference SNR by Γ_e.

Now get the other correlation coefficient

$$\alpha = \frac{\sigma_{yv}^{\ 2}}{\sigma_y\cdot \sigma_v}$$

We have the numerator as

$$\sigma_{yv}^{\ 2} = \frac{1}{2}\cdot E\left(y(k\cdot T)\cdot \overline{v(k\cdot T)}\right) = \frac{1}{2}\cdot E\left[\left(\sqrt{2\cdot E_b}\cdot g(k\cdot T)\cdot b(k) + n(k\cdot T)\right)\cdot \overline{(g(k\cdot T) + e(k\cdot T))}\right]$$

$$= \sqrt{2\cdot E_b}\cdot \sigma_g^{\ 2}\cdot b(k)$$

Therefore

$$\alpha = \frac{\sigma_{yv}^{\ 2}}{\sigma_y\cdot \sigma_v} = \frac{\sqrt{2\cdot E_b}\cdot \sigma_g^{\ 2}\cdot b(k)}{\sqrt{\left(2\cdot E_b\cdot \sigma_g^{\ 2} + N_o\right)\cdot\left(\sigma_g^{\ 2} + \sigma_e^{\ 2}\right)}}$$

$$\blacksquare = \frac{b(k)}{\sqrt{\left(1+\frac{N_o}{2\cdot E_b\cdot\sigma_g^{\;2}}\right)\cdot\left(1+\frac{\sigma_e^{\;2}}{\sigma_g^{\;2}}\right)}} = \frac{b(k)}{\sqrt{\left(1+\Gamma_b^{\;-1}\right)\cdot\left(1+\Gamma_r^{\;-1}\right)}}$$

or

$$\alpha = \frac{b(k)\cdot\rho}{\sqrt{\left(1+\Gamma_b^{\;-1}\right)}}$$

and it depends on $b(k)$.

(b) The decision variable is

$$q(k) = \mathrm{Re}\left(y(k\cdot T)\cdot\overline{v(k\cdot T)}\right) = \frac{1}{2}\cdot\left(y(k\cdot T)\cdot\overline{v(k\cdot T)} + \overline{y(k\cdot T)}\cdot v(k\cdot T)\right)$$

Define the arrays

$$\mathbf{z}(k) = \begin{pmatrix} y(k\cdot T) \\ v(k\cdot T) \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $q(k)$ is the quadratic form

$$q(k) = \frac{1}{2}\cdot\overline{\mathbf{z}(k)}^{T}\cdot\mathbf{Q}\cdot\mathbf{z}(k)$$

(c) To get the BER, assume that $b(k)=1$, so that an error is made if $q(k)<0$. From the last page of Section 4.1 in the notes, we have the probability that the decision variable is negative as

$$P_e = \Pr(q(k) < 0) = \frac{1}{2}\cdot(1-\alpha)$$

From the expression for α in part (a), with $b(k)=1$,

$$P_e = \frac{1}{2}\cdot\left[1-\frac{\rho}{\sqrt{\left(1+\Gamma_b^{\;-1}\right)}}\right] = \frac{1}{2}\cdot\left[1-\frac{1}{\sqrt{\left(1+\Gamma_b^{\;-1}\right)\cdot\left(1+\Gamma_r^{\;-1}\right)}}\right]$$

Note the error floor as Γ_b becomes infinite: P_e => 0.5(1-ρ), so good estimates are important.

(d) The BER for coherent detection is obtained by letting the reference SNR go to infinity (equivalently, letting ρ=1) in the result from part (c). We get

$$P_e = \frac{1}{2}\cdot\left[1 - \frac{1}{\sqrt{\left(1+\Gamma_b^{-1}\right)}}\right] = \frac{1}{2}\cdot\left(1 - \sqrt{\frac{\Gamma_b}{1+\Gamma_b}}\right)$$ coherent

For differential detection, where we use the previous pulse as our reference, it is easy to show that the effect of channel noise in that estimate is to make

$$\rho = \frac{1}{\sqrt{1+\Gamma_b^{-1}}}$$ differential

so that the result from part (c) becomes

$$P_e = \frac{1}{2}\cdot\left(1 - \frac{1}{1+\Gamma_b^{-1}}\right) = \frac{1}{2\cdot\left(1+\Gamma_b\right)}$$ differential

As noted in the question sheet, these are well-known expressions.

(e) In coherent detection, we have

$y = A\cdot g\cdot b + n$ $\qquad$ $v = g$ $\qquad$ where the time index k has been dropped

so that

$q = \mathrm{Re}\left(y\cdot\bar{g}\right) = \mathrm{Re}\left[A\cdot\left(|g|\right)^2\cdot b + n\cdot\bar{g}\right] = A\cdot x\cdot b + \nu$ $\qquad$ where $\qquad$ $\nu = \mathrm{Re}\left(n\cdot\bar{g}\right)$

Now examine the statistics of ν. The noise n is Gaussian, with variance $\frac{1}{2}\cdot E\left[\left(|n|\right)^2\right] = N_o$

so that the variance of its real and imaginary parts separately are also N_o. When conditioned on g, the product $n\cdot\bar{g}$ remains Gaussian. The phase of g doesn't matter, because n already has uniformly distributed phase, so the only discernible effect is that the variance is scaled by $|g|^2$. Thus

$$\sigma_\nu^2 = N_o\cdot\left(|g|\right)^2 = N_o\cdot x$$

The next step is to observe that q is equivalent to a real binary decision system with additive Gaussian noise. If b=1, the probability is error is just what you learned in your first course in communications

$P_e(x) = Q\left(\frac{A\cdot x}{\sigma_\nu}\right) = Q\left(\sqrt{2\cdot\Gamma_b\cdot x}\right)$ $\qquad$ where $\qquad$ $Q(z) = \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_z^\infty e^{\frac{-\alpha^2}{2}}\,d\alpha$

The average error rate is just the expectation over the squared magnitude of the gain

$$P_{e\_av} = \int_0^{\infty} P_e(x) \cdot p_x(x)\, dx$$

so we need the pdf $p_x(x)$. We have already seen in the notes that *z* has an exponential distribution, and its mean is

$$E(z) = E\left[(|g|)^2\right] = 2 \cdot \sigma_g^{\;2} = 1$$

so that $p_x(x) = e^{-x}$ for $x \geq 0$

Now we "merely" have to evaluate

$$P_{e\_av} = \int_0^{\infty} P_e(x) \cdot p_x(x)\, dx = \int_0^{\infty} Q\left(\sqrt{2 \cdot \Gamma_b \cdot x}\right) \cdot e^{-x}\, dx$$

With the definition of *Q*, this is clearly a case for integration by parts. Unfortunately, working with *x* directly is a challenge. At some point, you have to make a change of variables to *w*=sqrt(*x*), and this is equivalent to working with the Rayleigh distributed *w*=|*g*|, instead of *x*=|*g*|^2. I should have flagged this more clearly in the question statement. In any case, after making the change, we have

$$P_{e\_av} = \int_0^{\infty} Q\left(\sqrt{2 \cdot \Gamma_b} \cdot w\right) \cdot e^{-w^2} \cdot 2 \cdot w\, dw$$

You can see the Rayleigh pdf of *w* here.

To integrate by parts, we identify $u = Q\left(\sqrt{2 \cdot \Gamma_b} \cdot w\right)$ $dv = 2 \cdot w \cdot e^{-w^2} \cdot dw$

From these, we obtain

$$du = \frac{-\sqrt{2 \cdot \Gamma_b}}{\sqrt{2 \cdot \pi}} \cdot e^{-\Gamma_b \cdot w^2} \qquad v = -e^{-w^2}$$

$$P_{e\_av} = \frac{1}{2} - \frac{\sqrt{2 \cdot \Gamma_b}}{\sqrt{2 \cdot \pi}} \cdot \int_0^{\infty} e^{-\left(\Gamma_b + 1\right) \cdot w^2}\, dw$$

We recognise the integral as one side of the Gaussian pdf, but it is missing the standard deviation scale factor. Rewrite as

$$P_{e\_av} = \frac{1}{2} - \frac{\sqrt{2 \cdot \Gamma_b}}{\sqrt{2 \cdot \left(\Gamma_b + 1\right)}} \cdot \frac{\sqrt{2 \cdot \left(\Gamma_b + 1\right)}}{\sqrt{2 \cdot \pi}} \cdot \int_0^{\infty} e^{-\left(\Gamma_b + 1\right) \cdot w^2}\, dw$$

which simplifies to

$$P_{e\_av} = \frac{1}{2}\cdot\left(1 - \sqrt{\frac{\Gamma_b}{\Gamma_b + 1}}\right)$$

This is the same expression we obtained through use of quadratic forms.

A final note on Problem 2 - it is not usual for the estimation error variance $\sigma_e^{\ 2}$ to be independent of SNR. Normally, the channel estimation procedure is affected by the additive noise, so that increasing SNR also decreases the estimation error.