DEMONSTRATION OF TRIANGULARIZATION BY CHOLESKY AND GAUSSIAN ELIMINATION

This worksheet reduces a Hermition matrix to triangular factors by Cholesky and by the simpler Gaussian elimination, and shows numerically that they are equivalent.

Generate a Random Hermitian Matrix

$N := 5$ $\quad i := 0 .. N-1$ $\quad j := 0 .. N-1$

$X_{i,j} := \mathrm{rnd}(2) - 1$ <=== click once here, press F9, to get new matrix

$\mathbf{R} := \overline{\left(X^T\right)} \cdot X$ make it Hermitian

Here it is:

$$\mathbf{R} = \begin{bmatrix} 3.385 & 0.344 & -0.223 & 0.458 & 0.533 \\ 0.344 & 1.975 & 0.638 & -0.334 & -0.191 \\ -0.223 & 0.638 & 1.938 & -0.301 & 0.208 \\ 0.458 & -0.334 & -0.301 & 1.088 & 0.364 \\ 0.533 & -0.191 & 0.208 & 0.364 & 1.482 \end{bmatrix}$$

Cholesky Factorization

$\mathbf{F} := \mathrm{cholesky}(\mathbf{R})$

F is lower triangular

$$\mathbf{F} = \begin{bmatrix} 1.84 & 0 & 0 & 0 & 0 \\ 0.187 & 1.393 & 0 & 0 & 0 \\ -0.121 & 0.474 & 1.303 & 0 & 0 \\ 0.249 & -0.273 & -0.108 & 0.97 & 0 \\ 0.29 & -0.176 & 0.251 & 0.279 & 1.107 \end{bmatrix}$$

and it factors **X**:

$$\mathbf{F}\cdot\mathbf{F}^{T} - \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Its inverse is also lower triangular

$$\mathbf{F}^{-1} = \begin{bmatrix} 0.544 & 0 & 0 & 0 & 0 \\ -0.073 & 0.718 & 0 & 0 & 0 \\ 0.077 & -0.261 & 0.767 & 0 & 0 \\ -0.151 & 0.173 & 0.086 & 1.031 & 0 \\ -0.133 & 0.13 & -0.195 & -0.26 & 0.903 \end{bmatrix}$$

and it diagonalizes the noise covariance matrix

$$\mathbf{F}^{-1}\cdot\mathbf{R}\cdot\mathbf{F}^{-1^{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Gaussian Elimination

To factor **R** by Gaussian elimination, we augment it with the identity matrix, then do row reductions. Mathcad's column-oriented notation makes it easier to perform Gaussian elimination on the transpose, so it's column reductions.

$$\text{Gauss}(\mathbf{R}) := \left| \begin{array}{l} \mathbf{Z} \leftarrow \text{augment}(\mathbf{R}, \text{identity}(N))^T \\ \text{for } i \in 0 .. N-2 \\ \quad \text{for } j \in i+1 .. N-1 \\ \qquad \mathbf{Z}^{\langle j \rangle} \leftarrow \mathbf{Z}^{\langle j \rangle} - \dfrac{\mathbf{Z}_{i,j}}{\mathbf{Z}_{i,i}} \cdot \mathbf{Z}^{\langle i \rangle} \\ \mathbf{Z}^T \end{array} \right.$$

Upper and lower factors are then extracted by

$\text{temp} := \text{Gauss}(\mathbf{R})$

$\mathbf{U} := \text{submatrix}(\text{temp}, 0, N-1, 0, N-1)$ $\quad$ $\mathbf{L} := \text{submatrix}(\text{temp}, 0, N-1, N, 2 \cdot N-1)$

They are

$$\mathbf{U} = \begin{bmatrix} 3.385 & 0.344 & -0.223 & 0.458 & 0.533 \\ 0 & 1.94 & 0.661 & -0.38 & -0.246 \\ 0 & 0 & 1.699 & -0.141 & 0.327 \\ 0 & 0 & 0 & 0.94 & 0.271 \\ 0 & 0 & 0 & 0 & 1.227 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.102 & 1 & 0 & 0 & 0 \\ 0.101 & -0.341 & 1 & 0 & 0 \\ -0.147 & 0.168 & 0.083 & 1 & 0 \\ -0.147 & 0.144 & -0.216 & -0.288 & 1 \end{bmatrix}$$

and they satisfy $\quad \mathbf{L} \cdot \mathbf{R} = \mathbf{U} \quad$ as seen below

$$\mathbf{L}\cdot\mathbf{R} - \mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The effect of applying **L** to the noise covariance matrix is to whiten, but not normalize, the noise components:

$$\mathbf{L}\cdot\mathbf{R}\cdot\mathbf{L}^T = \begin{bmatrix} 3.385 & 0 & 0 & 0 & 0 \\ 0 & 1.94 & 0 & 0 & 0 \\ 0 & 0 & 1.699 & 0 & 0 \\ 0 & 0 & 0 & 0.94 & 0 \\ 0 & 0 & 0 & 0 & 1.227 \end{bmatrix}$$

It is easy to show (and to see) that this matrix is just the diagonal of **U** above. Thus the SNR on each bit decision, assuming correct decisions fed back, is

$\gamma_i := \mathbf{U}_{i,i}$ assuming unit noise variance

Now compare with Cholesky. Because it normalizes, as well as whitens, the noise, the SNR on each decision is the square of the diagonal entries of **F** (again assuming unit noise variance).

$$\gamma ch_i := \left(\mathbf{F}_{i,i}\right)^2$$

They are the same

$$\gamma^T = (3.385 \quad 1.94 \quad 1.699 \quad 0.94 \quad 1.227)$$

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