APPENDIX B

COMPLEX ENVELOPE NOTATION

1. Introduction

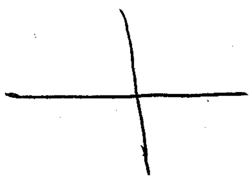
- Bandpass signals are ones with a null at and around DC;
  - narrowband $\Rightarrow$ typically, bandwidth < 10% centre freq.
  - wideband $\Longrightarrow$ typ., bandwidth > 20% centre frequency

The complex envelope is a convenient notation for them.

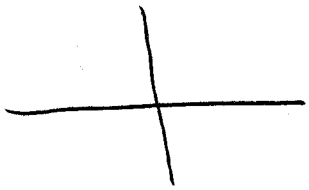
- The information is in the envelope (amplitude and phase), not the carrier. Distinguish between them by

$$\underbrace{\tilde{v}(t)}_{\text{bandpass, real}} = \mathrm{Re}\Big[\underbrace{v(t)}_{\text{lowpass, complex}}\ \underbrace{e^{j2\pi f_c t}}_{\text{complex carrier}}\Big]$$

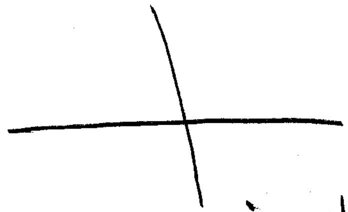
The complex envelope is a "time varying phasor".



AM



PM, FM



random noise at IF filter output

- Equivalent representations: if the complex envelope is

$$v(t) = x(t) + j\,y(t) = A(t)\,e^{j\varphi(t)}$$

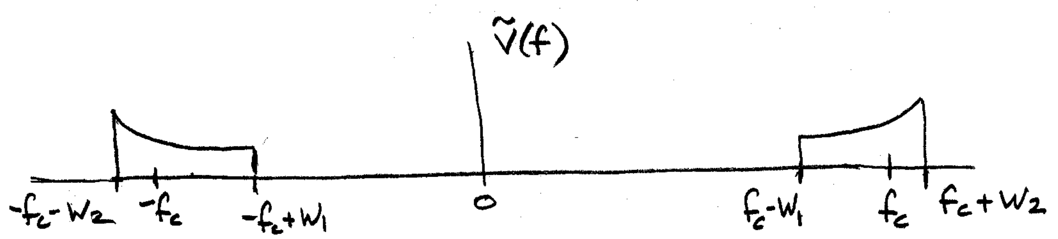
then

$$\tilde{v}(t) = x(t)\cos(2\pi f_c t) - y(t)\sin(2\pi f_c t)$$

$$= A(t)\cos\big(2\pi f_c t + \varphi(t)\big)$$

2. Generality of Complex Envelope

- Any real, bandpass signal can be written in complex envelope form. Consider this one:



"carrier" f_c not necessarily the centre

Because $\tilde{v}(t)$ is real, $\tilde{V}(f) = \tilde{V}^*(-f)$ i.e. conjugate symmetric.

Therefore

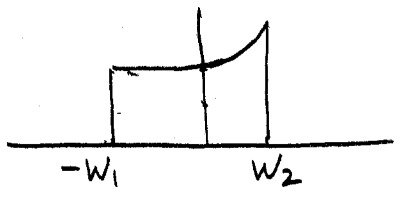
$$\tilde{v}(t) = \int_{-\infty}^{\infty} \tilde{V}(f) e^{j2\pi f t} df = 2\,\mathrm{Re}\left[\int_{f_c-W_1}^{f_c+W_2} \tilde{V}(f) e^{j2\pi f t} df\right]$$

$$= \mathrm{Re}\Big[e^{j2\pi f_c t} \underbrace{\int_{-W_1}^{W_2} 2\,\tilde{V}(\alpha+f_c) e^{j2\pi \alpha t} d\alpha}_{v(t),\ \text{lowpass, the complex envelope}}\Big]$$

$$= \mathrm{Re}\left[v(t)\, e^{j2\pi f_c t}\right]$$

and the Fourier transform of $v(t)$ is just a frequency translate, doubled:

$$V(f) = 2\,\tilde{V}_+(f+f_c)$$ (subscript + means positive frequencies only)



(or $V(f) = 2\,\tilde{V}_-^*(f-f_c)$)

- Notes:
  - Choice of "centre" frequency is arbitrary. Changes in f_c value are absorbed in the complex envelope.

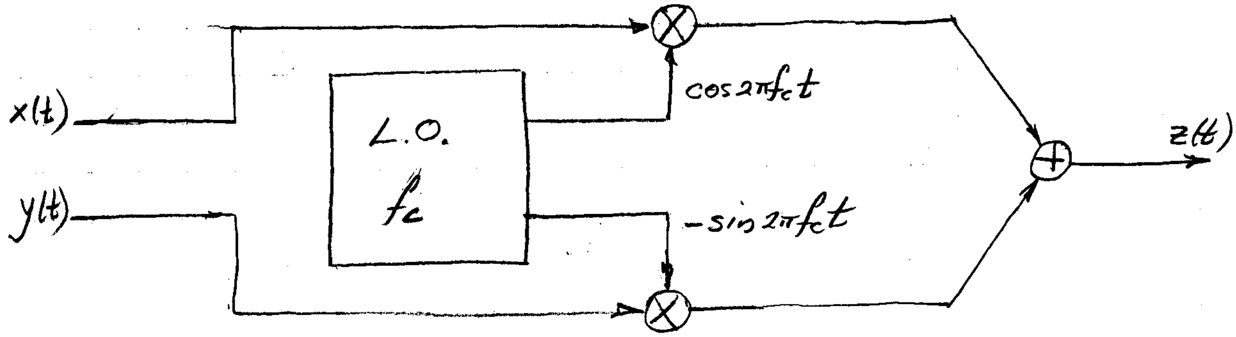
$$\tilde{v}(t) = \mathrm{Re}\left[v(t)\, e^{j2\pi f_c t}\right] = \mathrm{Re}\Big[\underbrace{v(t)\, e^{-j2\pi\Delta t}}_{\text{new complex envelope}}\ \underbrace{e^{j2\pi(f_c+\Delta)t}}_{\text{shifted centre}}\Big]$$

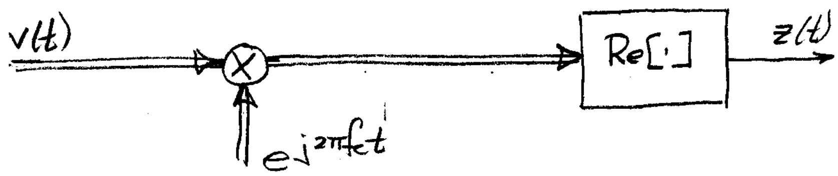
  - The transform $V(f)$ of the complex envelope is not necessarily symmetric about 0 Hz.

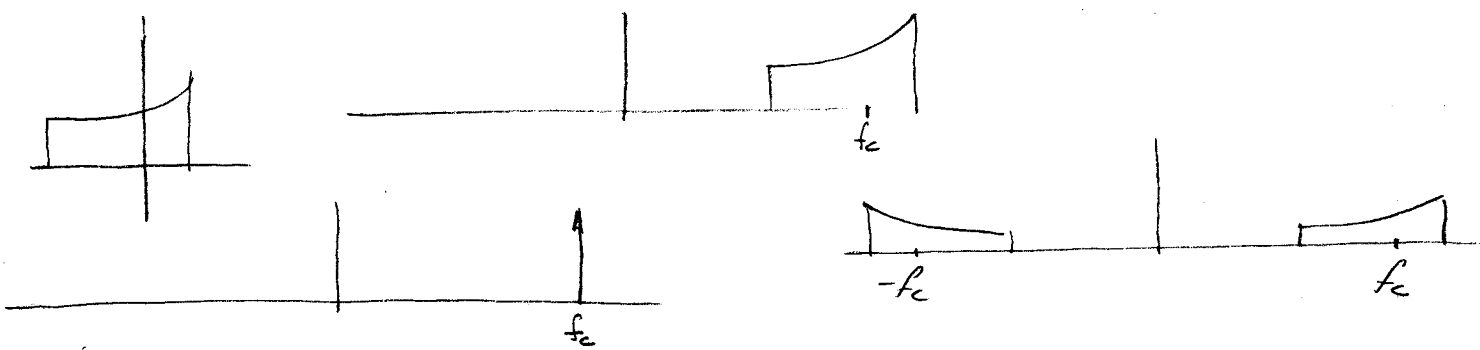
  - $A(t)$ and $\phi(t)$ are amplitude and phase w.r.t. an unmodulated carrier. The complex carrier $e^{j2\pi f_c t}$ itself has amplitude 1, phase $2\pi f_c t$.

  - The transform $V(f)$ is twice as large as either component of $\tilde{V}(f)$.

2.1.3 Generation of Band Pass Signals by Complex Envelope







Notes: - x(t), y(t) generated by modem or speech processing

- has to be balanced:
  - 90° separation or crosstalk
  - same gain each branch or distortion (e.g. const env FM becomes non const env; e.g. SSB generation with $y(t) = \mathcal{H}[x(t)]$ has imperfect sideband cancellation.

2.1.4 Recovery of Complex Envelope

Receive real b.p. signal $\mathrm{Re}[r(t)e^{j2\pi f_c t}]$

Multiply by $2\cos 2\pi f_c t = \mathrm{Re}[2e^{j2\pi f_c t}]$

Fact: for complex α, β $\quad \mathrm{Re}[\alpha]\,\mathrm{Re}[\beta] = \frac{1}{2}\mathrm{Re}[\alpha\beta^* + \alpha\beta]$

So $\mathrm{Re}[r(t)e^{j2\pi f_c t}]\,\mathrm{Re}[2e^{j2\pi f_c t}]$

$$= \frac{1}{2}\mathrm{Re}[\underbrace{2r(t)}_{\text{lopass}} + \underbrace{2r(t)e^{j4\pi f_c t}}_{\text{double freq}}]$$

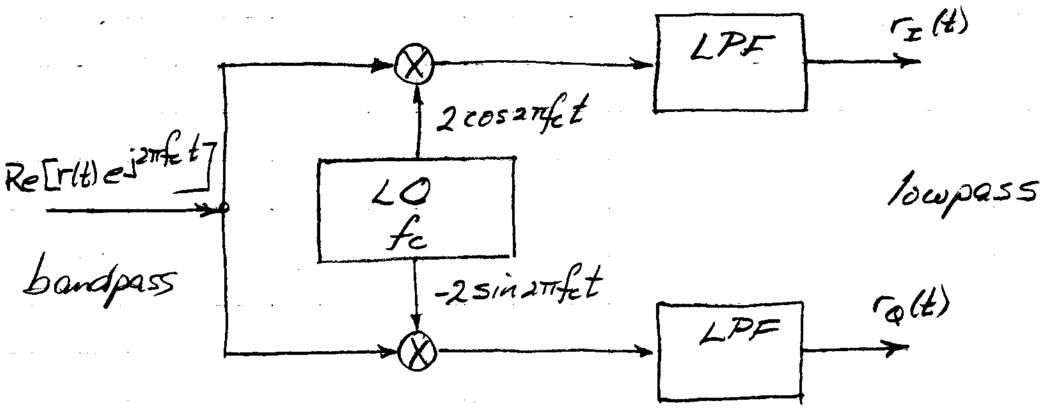
and lopass to get $\mathrm{Re}[r(t)] = r_I(t) = x(t)$

Similarly $-2\sin 2\pi f_c t = \mathrm{Re}[2j\,e^{j2\pi f_c t}]$

so $\mathrm{Re}[r(t)e^{j2\pi f_c t}]\,\mathrm{Re}[2j\,e^{j2\pi f_c t}]$

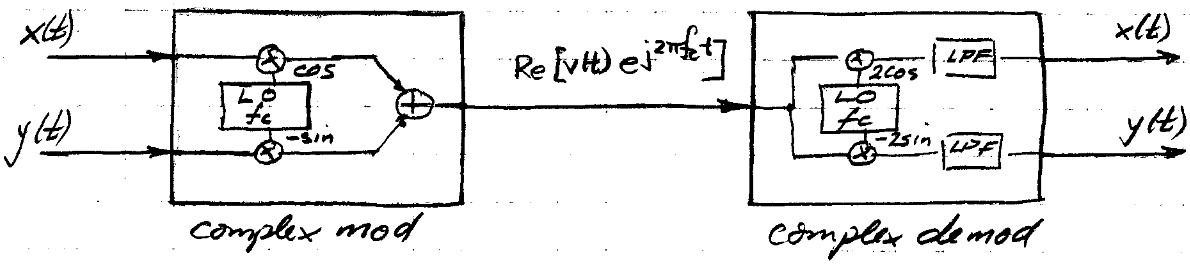
$$= \frac{1}{2}\mathrm{Re}[-2j\,r(t) + 2j\,r(t)e^{j4\pi f_c t}]$$

and lopass to get $\mathrm{Re}[-j\,r(t)] = \mathrm{Im}[r(t)] = r_Q(t) = y(t)$



Notes: - complex env phase is wrt $\underline{\underline{Rx}}$ L.O.

example quadrature multiplexing — 2 independent signals in the same bandwidth.



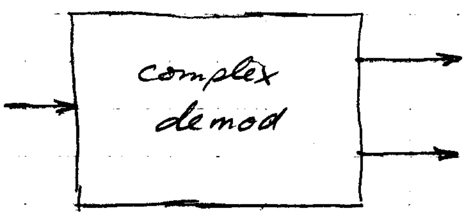
Tx $v(t) = x(t) + jy(t)$

Rx $r(t) = v(t)$

- if the channel imparts a phase shift, or if Tx and Rx LO's are not locked, then get phase difference θ
- arbitrarily ascribe it to the channel:

$$r(t) = v(t)\, e^{j\theta}$$

- crosstalk upon demodulation:

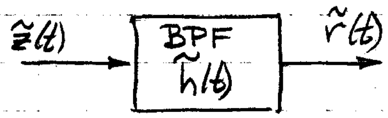


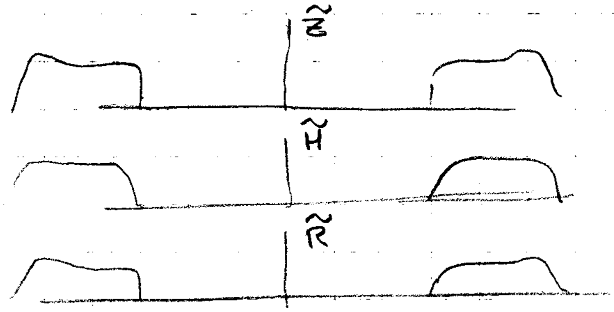
$$\mathrm{Re}[r(t)] = x(t)\cos\theta - y(t)\sin\theta$$

$$\mathrm{Im}[r(t)] = x(t)\sin\theta + y(t)\cos\theta$$

2.1.5 Bandpass Filtering

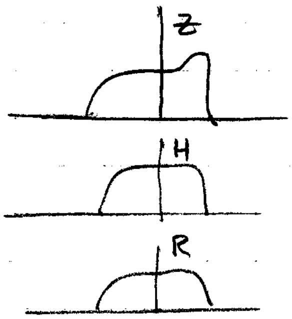
- Adopt notation $\tilde{w}(t)$ is real bp signal, $w(t)$ is its complex env





- Clearly $\tilde{R}(f) = \tilde{Z}(f)\tilde{H}(f)$

- so $R(f) = Z(f)\,H(f)$



$Z(f) = 2\,\tilde{Z}_+(f+f_c)$

$H(f) = \tilde{H}_+(f+f_c)$

$R(f) = 2\,\tilde{R}_+(f+f_c)$

- time domain proof:

$$\tilde{r}(t) = \tilde{z}(t) \circledast \tilde{h}(t) = \int \tilde{h}(\alpha)\,\tilde{z}(t-\alpha)\,d\alpha$$

$$= \int \tilde{h}(\alpha)\,\mathrm{Re}\left[z(t-\alpha)\,e^{j2\pi f_c(t-\alpha)}\right] d\alpha$$

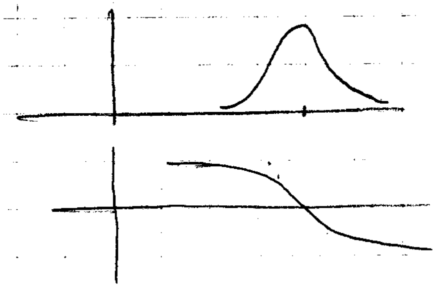
$$= \mathrm{Re}\left[e^{j2\pi f_c t}\underbrace{\int \tilde{h}(\alpha)\,z(t-\alpha)\,e^{-j2\pi f_c \alpha}\,d\alpha}_{r(t)}\right]$$

$$r(t) = \int h(\alpha)\,z(t-\alpha)\,d\alpha$$

where $h(t) = \tilde{h}(t)\,e^{-j2\pi f_c t}\Big|_{\text{lopass}} = h_+(t)\,e^{-j2\pi f_c t}$ left shifted

(lopass (since $z(t)$ is lopass))

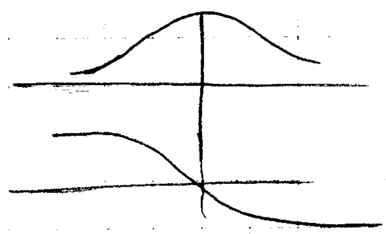
<u>example</u> 2^{nd} order tuned filter looks like 1^{st} order to envelope of narrowband modulation



$$H(f) = \frac{1}{1+jQ\left(u-\frac{1}{u}\right)} \qquad \text{where } u = f/f_0$$

Near resonance $u = 1+\delta \qquad |\delta| \ll 1$

$$H(f) = \frac{1}{1+jQ\left(1+\delta-\frac{1}{1+\delta}\right)} \approx \frac{1}{1+j2Q\delta} = \frac{1}{1+j\frac{2Q}{f_0}(f-f_0)}$$

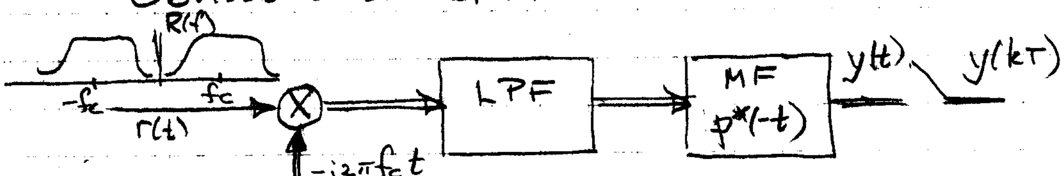


$$H_+(f+f_0) = \frac{1}{1+j\frac{2Q}{f_0}f}$$

a 1^{st} order LPF

<u>example</u> Bandpass matched filter. Real carrier signal comes in.

Obvious demodulator:



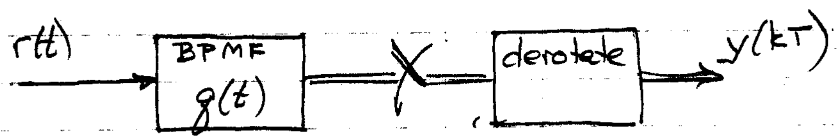
This is equivalent to a complex bandpass matched filter:

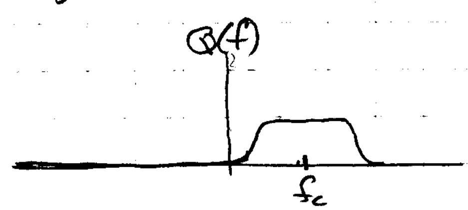
$$y(t) = \left(r(t)\, e^{-j2\pi f_c t}\right) \circledast p^*(-t)$$

$$y(kT) = \int r(kT-\alpha)\, e^{-j2\pi f_c(kT-\alpha)}\, p^*(-\alpha)\, d\alpha$$

$$= e^{-j2\pi f_c Tk} \int r(kT+\beta)\, p^*(\beta)\, e^{-j2\pi f_c \beta}\, d\beta \qquad (\beta = -\alpha)$$

$$= e^{-j2\pi f_c Tk} \left(r(t) \circledast g(t)\right)\Big|_{t=kT} \qquad g(t) = p(t)\, e^{j2\pi f_c t}$$





Combines demodulation with matched filter.

2.1.6 Fourier Symmetry Relations

Now that we're committed to complex time functions, better review Fourier symmetries. If $v(t) \longleftrightarrow V(f)$ then:

① $V(f) = \int_{-\infty}^{\infty} v(t)\, e^{-j2\pi f t}\, dt \;;\quad v(t) = \int_{-\infty}^{\infty} V(f)\, e^{j2\pi f t}\, df$

② $v(-t) \longleftrightarrow V(-f)$ time/freq reversal

③ $v^*(t) \longleftrightarrow V^*(-f) \;;\; V^*(f) \longleftrightarrow v^*(-t)$

④ $v(t)$ real $\Longleftrightarrow$ $V(-f) = V^*(f)$ conjugate symmetry

⑤ $v(t)$ imag $\Longleftrightarrow$ $V(-f) = -V^*(f)$ conjugate odd symmetry

⑥ $v(t)$ real and even $\Longleftrightarrow$ $V(f)$ real and even

⑦ if $v(t) = x(t) + j\,y(t)$ $\quad$ ($x(t)$, $y(t)$ real)

$$\mathrm{Re}[v(t)] \longleftrightarrow X(f) = \tfrac{1}{2}\left[V(f) + V^*(-f)\right]$$

$$\mathrm{Im}[v(t)] \longleftrightarrow Y(f) = \tfrac{1}{2j}\left[V(f) - V^*(-f)\right]$$

⑧ $v(t)\, e^{j2\pi f_0 t} \longleftrightarrow V(f - f_0)$ complex mod (freq shift)

⑨ $v(t) \circledast u^*(t) = \int v(\alpha)\, u^*(t-\alpha)\, d\alpha \longleftrightarrow V(f)\, U^*(f)$ convolution

⑩ $v(t) \circledast u^*(-t) = \int v(\alpha)\, u^*(\alpha - t)\, d\alpha \longleftrightarrow V(f)\, U^*(-f)$ cross correlation

⑪ $\int v(\alpha)\, u^*(\alpha)\, d\alpha = \int V(f)\, U^*(f)\, df$ Parseval

2.1.7 Relation to Hilbert Transform

$$s(t) \longrightarrow \boxed{-j\,\mathrm{sgn}(f)} \longrightarrow \hat{s}(t)$$

Hilbert transform filter

$H(f) = -j\,\mathrm{sgn}(f)$ (conj symm)

$h(t) = 2/t, \quad 0 < |t| < \infty$ (real)

$$\hat{S}(f) = -j\,\mathrm{sgn}(f)\,S(f) \;; \qquad \hat{s}(t) = 2 \fint_{-\infty}^{\infty} \frac{s(\alpha)}{t-\alpha}\,d\alpha$$

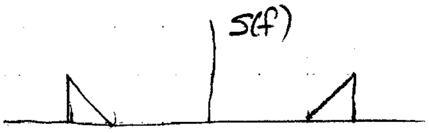
Note $\hat{s}(t)$ is real if $s(t)$ is.

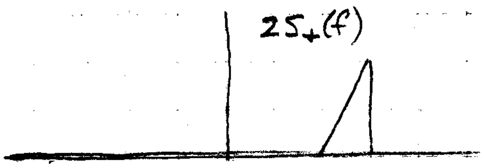
See DSP literature for approx Hilbert transformers.

- Consider bandpass $s(t)$:

$$s(t) + j\,\hat{s}(t) \longleftrightarrow S(f) + j\,\hat{S}(f) = 2\,S_+(f)$$

$$= V(f - f_c)$$





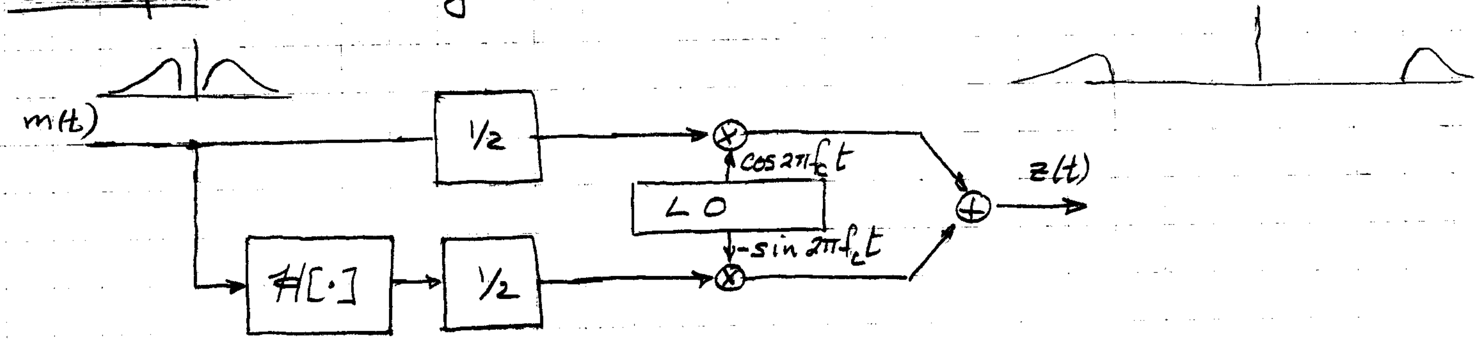
so $v(t)\,e^{j2\pi f_c t} = s(t) + j\,\hat{s}(t)$

- $\mathcal{H}[\;]$ is a wideband -90° phase shifter:

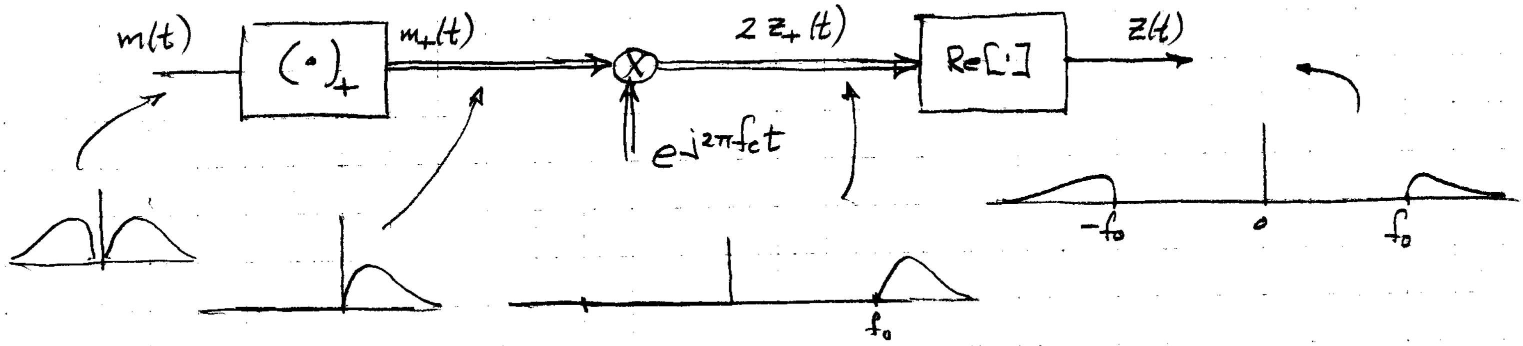
$$s(t) \longrightarrow \boxed{-90^\circ} \longrightarrow \hat{s}(t)$$

$$s(t) = \mathrm{Re}\left[v(t)\,e^{j2\pi f_0 t}\right] \;; \qquad \hat{s}(t) = \mathrm{Re}\left[-j\,v(t)\cdot e^{j2\pi f_0 t}\right]$$

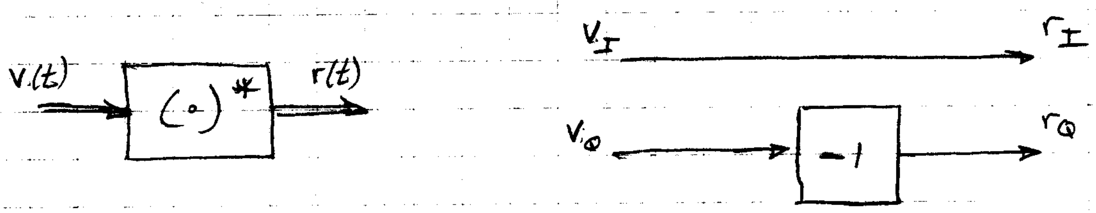
example SSB generation:

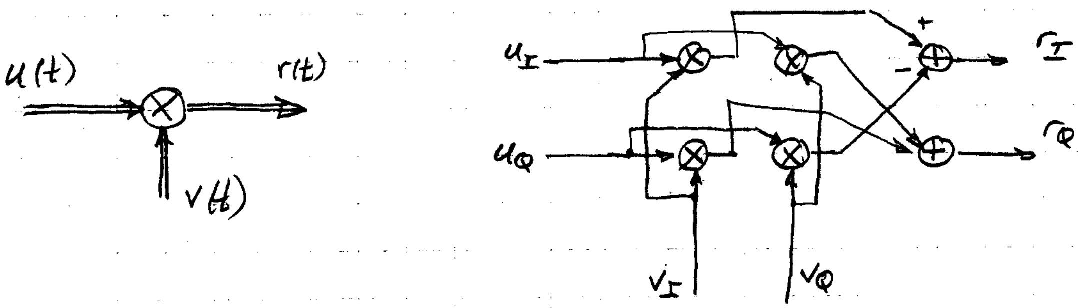


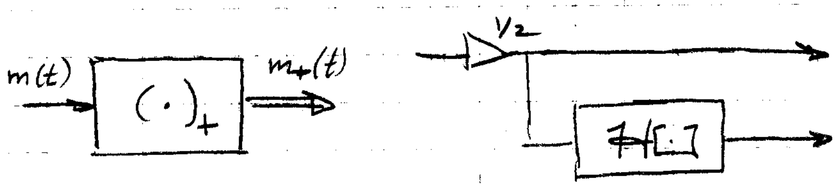
symbolically:

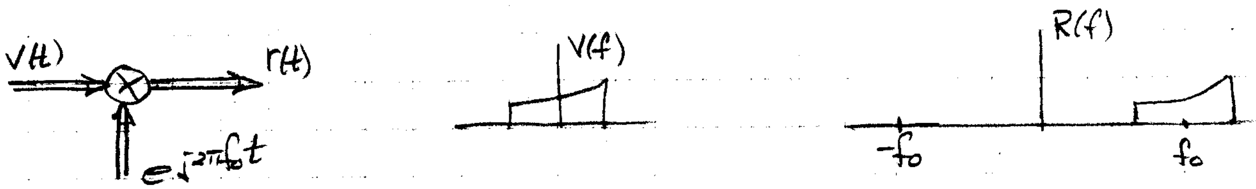


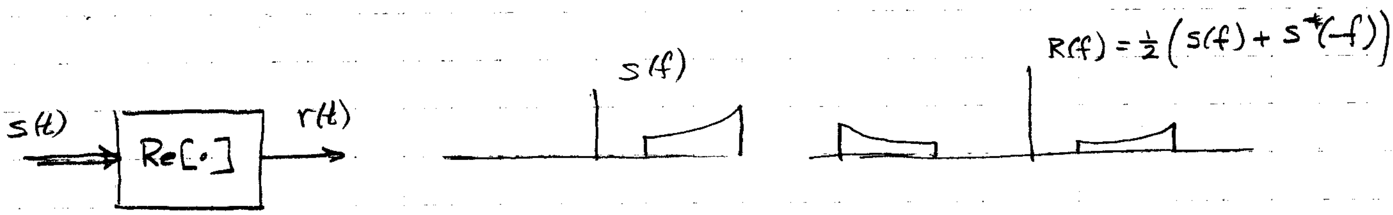
2.1.8 Building Blocks

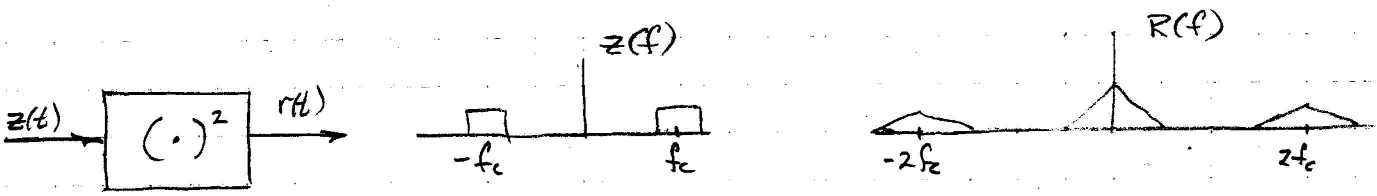










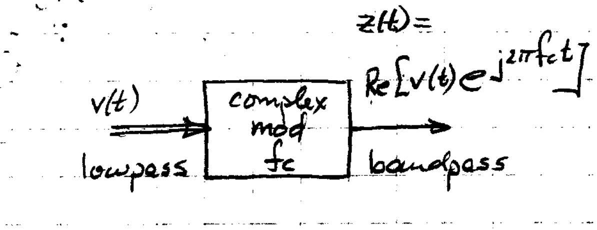


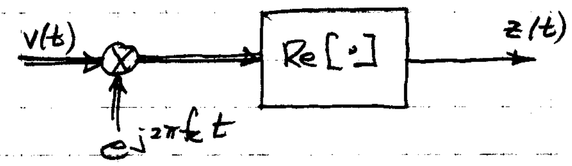
$$r(t) = z^2(t) = \mathrm{Re}\left[v(t)e^{j2\pi f_c t}\right]\,\mathrm{Re}\left[v(t)e^{j2\pi f_c t}\right]$$

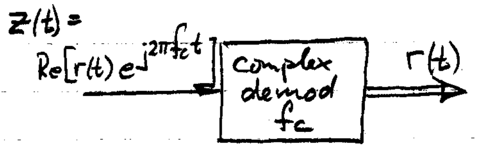
$$= \tfrac{1}{2}\,\mathrm{Re}\left[|v(t)|^2 + v^2(t)\,e^{j4\pi f_c t}\right]$$

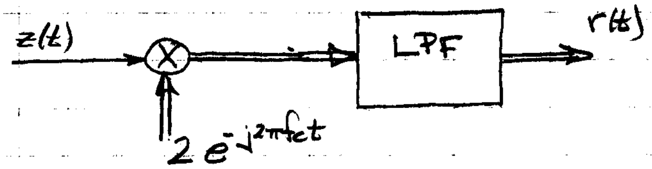
if PM or FM $v(t) = Ae^{j\varphi(t)}$ so $v^2(t) = A^2 e^{j2\varphi(t)}$

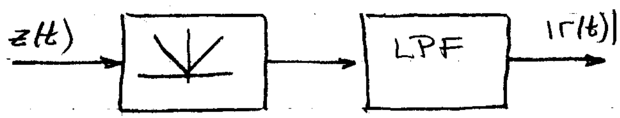
doubler doubles deviation, too.

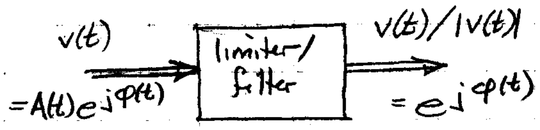


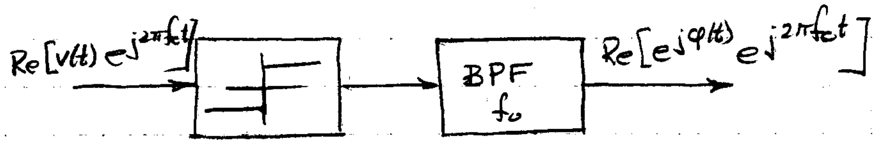












(details later)