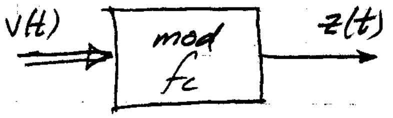
2.2 Bandpass Random Processes

2.2.1 Modulation

Suppose we form a bandpass process by upconverting a complex lowpass process $v(t) = x(t) + j\, y(t)$.



How do the 2nd order stats of $z(t)$ depend on those of $v(t)$?

$$R_z(t, t-\tau) = E\left[z(t)\, z(t-\tau)\right] = E\left[Re\left[v(t) e^{j2\pi f_c t}\right] Re\left[v(t-\tau) e^{j2\pi f_c (t-\tau)}\right]\right]$$

$$= \frac{1}{2} E\left[Re\left[\underbrace{v(t)\, v^*(t-\tau)\, e^{j2\pi f_c \tau}}_{\text{WSS if } v(t) \text{ WSS}} + \underbrace{v(t)\, v(t-\tau)\, e^{j2\pi f_c (2t-\tau)}}_{\text{dep on } t\,!} \right]\right]$$

Under what conditions does 2nd term vanish?
Expand it:

$$\left\{ x(t)\,x(t-\tau) - y(t)\,y(t-\tau) + j\left(x(t)\,y(t-\tau) + x(t-\tau)\,y(t) \right) \right\} e^{j2\pi f_c (2t-\tau)}$$

$$= \left\{ R_x(\tau) - R_y(\tau) + j\left(R_{xy}(\tau) + R_{xy}(-\tau) \right) \right\} e^{j2\pi f_c (2t-\tau)}$$

- if $R_x(\tau) = R_y(\tau)$ and $R_{xy}(\tau) = -R_{xy}(-\tau)$ then it vanishes

- if, instead of $E[\cdot]$, we define $R_z(t, t-\tau)$ with a time average

$$R_z(t, t-\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} z(t)\, z(t-\tau)\, dt$$

  then 2nd term vanishes because $e^{j4\pi f_c t}$ averages to zero.

- if carrier has random phase θ $e^{j(2\pi f_c t + \theta)}$ then carrier factor in 2nd term causes average of 0:

$$E_\theta\left[e^{j2\pi f_c (2t-\tau)}\, e^{j2\theta} \right] = 0$$

- from this point on, we'll ignore it.

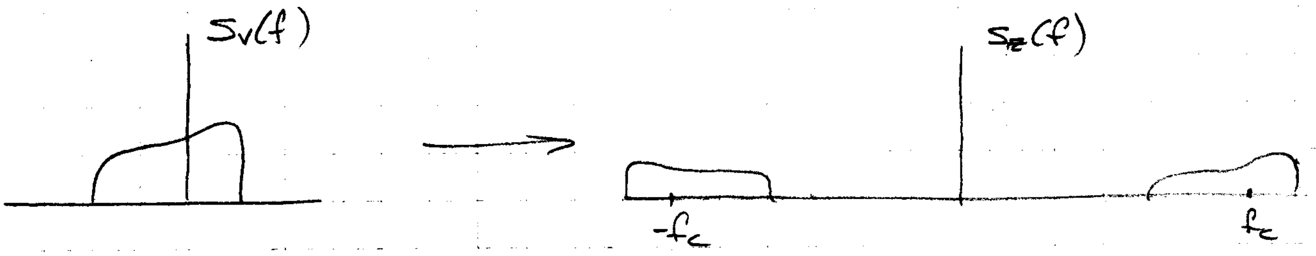
So $R_z(\tau) = \frac{1}{2} E\left[Re\left[v(t)v^*(t-\tau)e^{j2\pi f_c \tau}\right]\right]$

$$= Re\left[R_v(\tau)e^{j2\pi f_c \tau}\right]$$

where $R_v(\tau) = \frac{1}{2}\overline{v(t)v^*(t-\tau)}$

$$= \frac{1}{2}\left\{R_x(\tau) + R_y(\tau) + j\left(R_{xy}(-\tau) - R_{xy}(\tau)\right)\right\}$$

so $R_v(\tau)$ is the complex envelope of $R_z(\tau)$!



Why isn't $S_v(f)$ even ?? Is it real ??

- $S_v(f)$ is real, since $R_v(\tau)$ is conjugate symmetric:

  $R_v(\tau) = R_v^*(-\tau)$

- but $S_v(f)$ is symmetric (real and even) iff $R_v(\tau)$ is real:

  $R_{xy}(\tau) = R_{xy}(-\tau)$

  true if $x(t)$ $y(t)$ uncorrelated $R_{xy} \equiv 0$
  or identical $x(t) = y(t)$

- in fact, $Re[R_v(\tau)] \longleftrightarrow \frac{1}{2}\left[S_v(f) + S_v^*(-f)\right] = \frac{1}{2}\left[S_v(f) + S_v(-f)\right]$

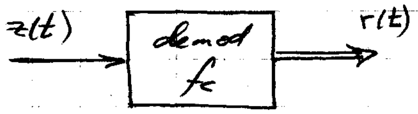
  $\frac{1}{2}\left(R_x(\tau) + R_y(\tau)\right) \longleftrightarrow \text{even}\left[S_v(f)\right]$

  $\frac{1}{2}\left(R_{xy}(-\tau) - R_{xy}(\tau)\right) \longleftrightarrow \text{odd}\left[S_v(f)\right]$

  so if $S_v(f)$ is not even, then $x(t)$ $y(t)$ correlated.

2.2.2 Demodulation

Given WSS bp process $z(t)$. What are the 2nd order stats of its complex envelope wrt $R_z(\tau)$?



$$R_z(t, t-\tau) = \overline{z(t)\, z(t-\tau)} = \tfrac{1}{2} \mathrm{Re}\Big[\overline{v(t) v^*(t-\tau)}\, e^{j2\pi f_c \tau} + \overline{v(t) v(t-\tau)}\, e^{j2\pi f_c (2t-\tau)} \Big]$$

Since $z(t)$ is WSS, $R_z(t, t-\tau)$ is a function only of τ. Hence:

$$R_z(\tau) = \mathrm{Re}\left[R_v(\tau)\, e^{j2\pi f_c \tau} \right] \qquad \text{as before}$$

Also $\overline{v(t)\, v(t-\tau)} \equiv 0$ for same reason

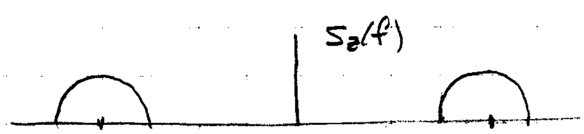
$$\therefore \quad R_x(\tau) \equiv R_y(\tau) \quad , \quad R_{xy}(\tau) = -R_{xy}(-\tau)$$

and

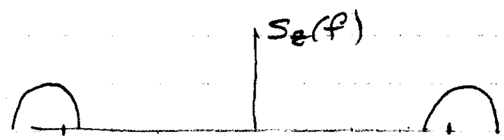
$$R_v(\tau) = \tfrac{1}{2}\big(R_x(\tau) + R_y(\tau)\big) + \tfrac{j}{2}\big(R_{xy}(-\tau) - R_{xy}(\tau) \big)$$

$$= R_x(\tau) - j\, R_{xy}(\tau)$$

This shows even more clearly that, unless $S_z(f)$ is symmetric about f_c, $x(t)$ and $y(t)$ are correlated

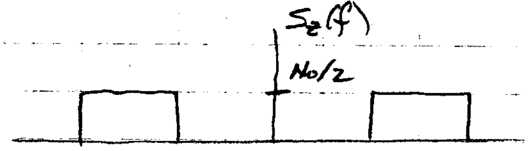


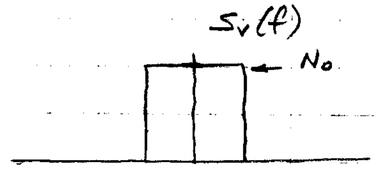
$R_{xy}(\tau) \equiv 0$



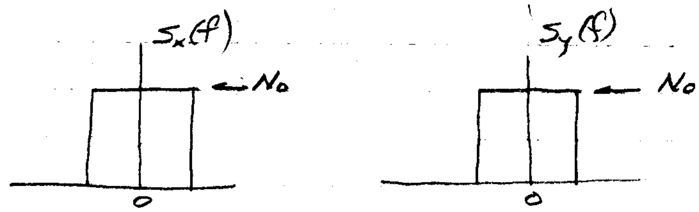
$R_{xy}(\tau) \not\equiv 0$

In particular, if $z(t)$ is bandlimited white noise with PSD equal to $N_0/2$





so $x(t)$, $y(t)$ are uncorrelated (indep if Gaussian), each with PSD N_0



1.2.3 Power Relations: Bandpass vs Complex Envelope

We have $\quad R_z(\tau) = Re\left[R_v(\tau)\, e^{j2\pi f_c \tau}\right]$

so $\quad P_z = P_v = \frac{1}{2}(P_x + P_y)$

i.e. modulation cuts the power in half.