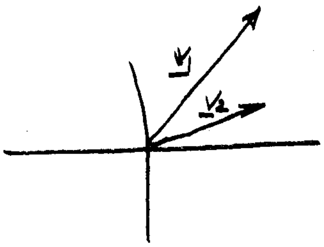
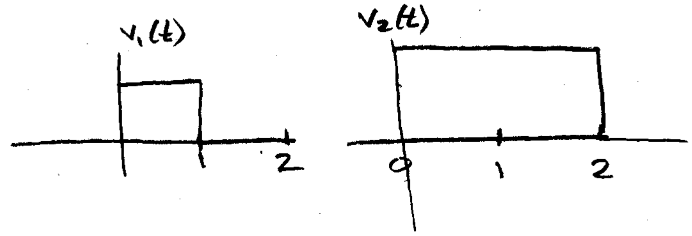
5.2 Orthogonal, Non-Orthogonal and Complete Bases

$\boxed{\text{P+S 7.1.3}}$

- A basis does not have to be orthonormal

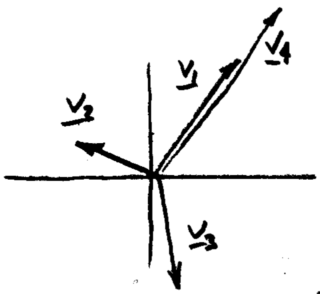


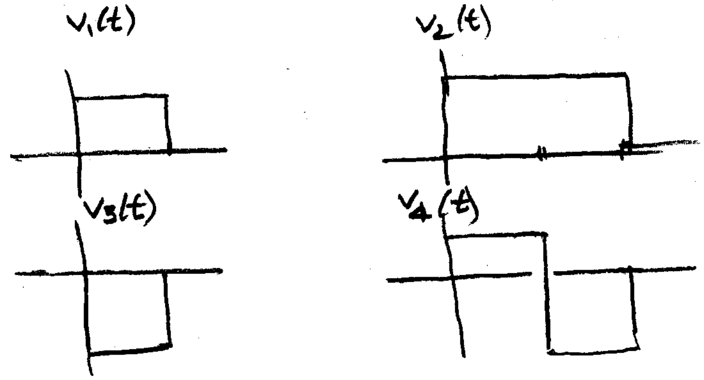


$\underline{v}_1, \underline{v}_2$ span this 2-D space

$v_1(t)$ and $v_2(t)$ span the same space as $\varphi_1(t), \varphi_2(t)$ pg 5.1.4

but the vectors should be linearly independent.





$\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4$ still span only 2D space. $\underline{v}_1, \underline{v}_4$ are colinear

The functions still span the same 2-D space. $v_1(t), v_3(t)$ are colinear.

- How do we determine the coefficients of a vector wrt an arbitrary basis?
  - Suppose the basis is

$$\{\underline{v}_i\} \qquad\qquad \{v_i(t)\}$$

and we want to represent some element of the space

$$\underline{x} = \sum_{i=1}^{N} a_i \underline{v}_i \qquad\qquad x(t) = \sum_{i=1}^{N} a_i v_i(t)$$

How to calculate the coefficients a_i?

– Take the inner product of $\underline{x}$ (or $x(t)$) and every basis vector

$$\underline{x} \cdot \underline{v}_k = \sum_{i}^{N} a_i \underline{v}_i \cdot \underline{v}_k \, , \quad \forall k, \quad \int x(t)\, v_k(t)\, dt = \sum_{i=1}^{N} a_i \int v_i(t)\, v_k(t)\, dt$$

Define

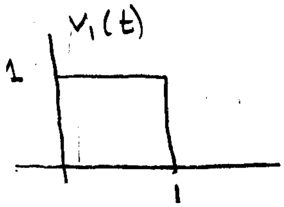
$$\underline{p} = \begin{bmatrix} (\underline{x}, \underline{v}_1) \\ (\underline{x}, \underline{v}_2) \\ \vdots \\ (\underline{x}, \underline{v}_N) \end{bmatrix} \qquad R = \begin{bmatrix} (\underline{v}_1, \underline{v}_1) & (\underline{v}_1, \underline{v}_2) & \cdots & (\underline{v}_1, \underline{v}_N) \\ (\underline{v}_2, \underline{v}_1) & (\underline{v}_2, \underline{v}_2) & & \vdots \\ \vdots & & \ddots & \\ (\underline{v}_N, \underline{v}_1) & \cdots & & (\underline{v}_N, \underline{v}_N) \end{bmatrix}$$

The equations are equivalent to

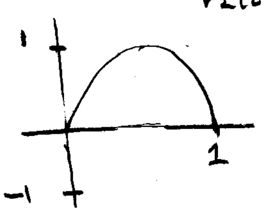
$$R\, \underset{\uparrow \text{ coefficients}}{\underline{a}} = \underline{p} \qquad \underline{a} = R^{-1} \underline{p}$$

example

$v_1(t)$

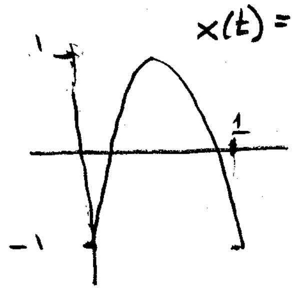


$v_2(t) = \sin(\pi t)$



$x(t) = -1 + 2\sin(\pi t)$

What are the coefficients of



$$R = \begin{bmatrix} \int v_1^2(t)\, dt & \int v_1(t) v_2(t)\, dt \\ \int v_2(t) v_1(t)\, dt & \int v_2^2(t) \end{bmatrix} \qquad \underline{p} = \begin{bmatrix} \int x(t) v_1(t)\, dt \\ \int x(t) v_2(t)\, dt \end{bmatrix} = \begin{bmatrix} \frac{4}{\pi} - 1 \\ -\frac{2}{\pi} + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2/\pi \\ 2/\pi & \frac{1}{2} \end{bmatrix}$$

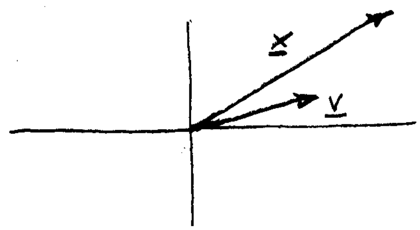
Solution is $\underline{a} = R^{-1}\underline{p} = \frac{1}{\pi^2 8}\begin{bmatrix} \pi^2 & -4\pi \\ -4\pi & 2\pi^2 \end{bmatrix}\begin{bmatrix} \frac{4}{\pi}-1 \\ -\frac{2}{\pi}+1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

so

$$x(t) = -v_1(t) + 2v_2(t)$$

- What if $\underline{x}$ does not lie completely in the space spanned by $\underline{v}_1$ to $\underline{v}_N$? Then we can't get equality $\underline{x} = \sum_i a_i \underline{v}_i$ so we minimize the error between $\underline{x}$ and its approximation $\hat{\underline{x}} = \sum_i a_i \underline{v}_i$; that is, choose the coefficients a_i to minimize the norm of $\underline{e} = \underline{x} - \hat{\underline{x}} = \underline{x} - \sum_i a_i \underline{v}_i$. Sketches help.

1-D basis:



$\hat{\underline{x}} = x\underline{v}$ lies in $\underline{v}$ direction

$|\underline{e}|^2 = |\underline{x} - x\underline{v}|^2$ is minimized by orthogonal projection

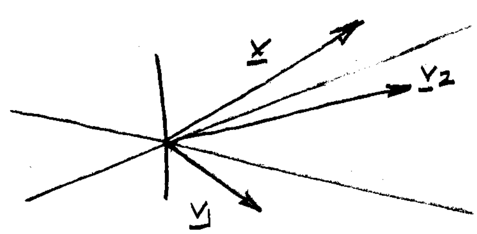
Pythagoras

$x =$

$\hat{\underline{x}} =$

$\underline{e} \cdot \underline{v} =$

2-D basis



$\underline{v}_1, \underline{v}_2$ lie in the plane

$\underline{x}$ has a component above the plane.

$\hat{\underline{x}}$ must lie in the plane

$|\underline{e}|^2 = |\underline{x} - \hat{\underline{x}}|^2$ is minimized if $\hat{\underline{x}}$ is the orthogonal projection of $\underline{x}$ onto the space spanned by $\underline{v}_1, \underline{v}_2$

$\underline{e} \cdot \hat{\underline{x}} =$ $\qquad$ $\underline{e} \cdot \underline{v}_1 =$ $\qquad$ $\underline{e} \cdot \underline{v}_2 =$

To obtain the coefficients, we minimize the squared norm of the error

$$J = \underline{e} \cdot \underline{e} = \left(\underline{x} - \sum_{i=1}^{N} a_i \underline{v}_i\right) \cdot \left(\underline{x} - \sum_{k=1}^{N} a_k \underline{v}_k\right)$$

$$\frac{\partial J}{\partial a_n} = 0 \Rightarrow -2\,\underline{e} \cdot \underline{v}_n = 0 \quad \text{for } n = 1..N$$ "orthogonality principle"

or

$$a_1\, \underline{v}_1 \cdot \underline{v}_n + a_2\, \underline{v}_2 \cdot \underline{v}_n + \cdots + a_N\, \underline{v}_N \cdot \underline{v}_n = \underline{x} \cdot \underline{v}_n \quad \text{for } n = 1..N$$

This is a set of linear equations

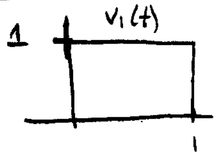
$$\begin{bmatrix} \underline{v}_1 \cdot \underline{v}_1 & \underline{v}_2 \cdot \underline{v}_1 & \cdots & \underline{v}_N \cdot \underline{v}_1 \\ \underline{v}_1 \cdot \underline{v}_2 & \underline{v}_2 \cdot \underline{v}_2 & & \\ & & \ddots & \\ \underline{v}_1 \cdot \underline{v}_N & & & \underline{v}_N \cdot \underline{v}_N \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} \underline{x} \cdot \underline{v}_1 \\ \underline{x} \cdot \underline{v}_2 \\ \vdots \\ \underline{x} \cdot \underline{v}_N \end{bmatrix}$$

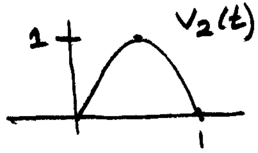
$$R\,\underline{a} = \underline{p}$$

By analogy, we have solved all these problems:

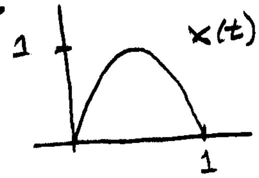
- minimum integrated squared error approximation of one function as a LC of others
- minimum sum of squared errors approx of one N-tuple as a LC of others
- minimum error variance approximation (i.e. estimation) of one random variable as a LC of others.

<u>example</u> use these basis functions





to approximate



with least integrated squared error.

$\hat{x}(t) = a_1 v_1(t) + a_2 v_2(t)$

First, the Gram matrix, needed regardless of choice of $x(t)$.

$$(v_1, v_1) = 1 \qquad (v_1, v_2) = \int_0^1 1 \cdot 4t(1-t)\,dt = \frac{2}{3}$$

$$(v_2, v_1) = (v_1, v_2) = \frac{2}{3} \qquad \|v_2\|^2 = \int_0^1 16t^2(1-t)^2\,dt = \frac{8}{15}$$

Next, the cross correlation vector

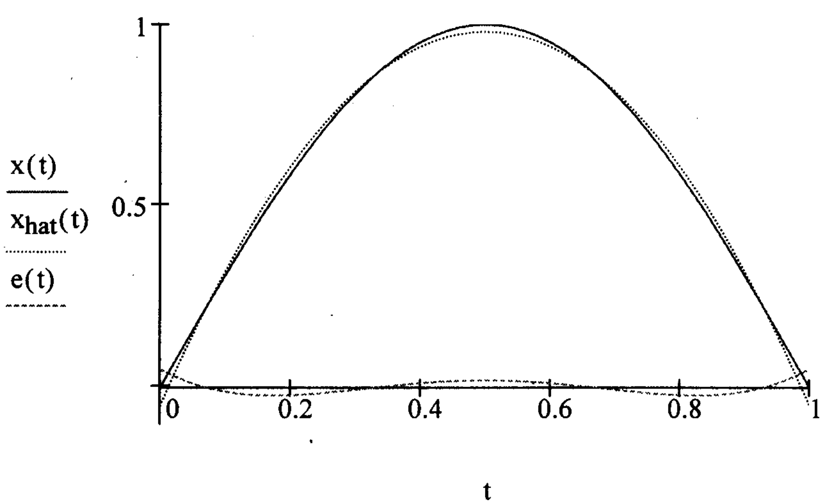
$$\int_0^1 x(t)\,v_1(t)\,dt = \int_0^1 \sin(\pi t)\,dt = \frac{2}{\pi}$$

$$\int_0^1 x(t)\,v_2(t)\,dt = \int_0^1 4t(1-t)\sin(\pi t)\,dt = \frac{16}{\pi^3}$$

To find best coefficients a_1, a_2, solve

$$\begin{bmatrix} 1 & 2/3 \\ 2/3 & 8/15 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\pi} \\ 16/\pi^3 \end{bmatrix}, \quad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 & -15/2 \\ -15/2 & 45/4 \end{bmatrix} \begin{bmatrix} \frac{2}{\pi} \\ \frac{16}{\pi^3} \end{bmatrix} = \begin{bmatrix} 12 \cdot \frac{\pi^2 - 10}{\pi^3} \\ -15 \cdot \frac{\pi^2 - 12}{\pi^3} \end{bmatrix}$$

$$\hat{x}(t) = 12\,\frac{\pi^2 - 10}{\pi^3} - 15 \cdot \frac{\pi^2 - 12}{\pi^3} \cdot 4t(1-t) = -0.05 + 1.124\,t(1-t)$$



orthogonality ?

Pythagoras ?

- Life is simpler if the basis set is orthonormal.

$$(\underline{\underline{v}}_i, \underline{\underline{v}}_k) = \delta_{ik}$$

Then Gram matrix becomes

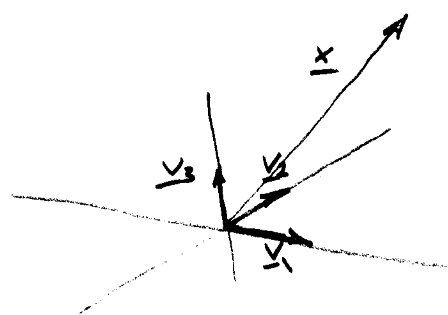
$$R = \begin{bmatrix} (\underline{\underline{v}}_1, \underline{\underline{v}}_1) & (\underline{\underline{v}}_1, \underline{\underline{v}}_2) & & \\ (\underline{\underline{v}}_2, \underline{\underline{v}}_1) & (\underline{\underline{v}}_2, \underline{\underline{v}}_2) & & \\ & & \ddots & \\ & & & (\underline{\underline{v}}_N, \underline{\underline{v}}_N) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} = I$$

Cross inner prod vector is still

$$\underline{p} = \begin{bmatrix} (\underline{\underline{x}}, \underline{\underline{v}}_1) \\ \vdots \\ (\underline{\underline{x}}, \underline{\underline{v}}_N) \end{bmatrix} \quad \text{so} \quad R\underline{a} = \underline{p} \text{ becomes } \underline{a} = \underline{p}$$

Geometric view:

$$\underline{x} = \underline{x}\cdot\underline{v}_1\ \underline{v}_1 + \underline{x}\cdot\underline{v}_2\ \underline{v}_2 + \underline{x}\cdot\underline{v}_3\ \underline{v}_3$$



<u>Example</u> Fourier series basis

$$v_k(t) = \frac{1}{\sqrt{T}} e^{j2\pi kt/T}, \quad 0 \le t \le T$$

Gram: $$(v_i(t), v_k(t)) = \int_0^T v_i(t) v_k^*(t)\,dt = \frac{1}{T}\int_0^T e^{j2\pi(i-k)t/T} = \delta_{ik}$$

cross: $$a_i = \int_0^T x(t) v_i^*(t)\,dt = \frac{1}{\sqrt{T}}\int_0^T x(t) e^{-j2\pi it/T} dt$$

approx: $-N \le i \le N$ $$\hat{x}_N(t) = \sum_{i=-N}^{N} a_i \frac{e^{j2\pi it/T}}{\sqrt{T}} = \sum_{i=-N}^{N} e^{j2\pi it/T} \frac{1}{T}\int_0^T x(t) e^{-j2\pi it/T} dt$$