The general interpretation of approximation in these inner product spaces tells us that:

- This choice of coefficients minimizes the energy (the "squared length") of the error
$$e_N(t) = x(t) - \hat{x}_N(t)$$
- The error is orthogonal to all the $v_i(t)$, $1 \le i \le N$ and consequently to $\hat{x}_N(t)$
$$\int_0^T e_N(t)\, v_i^*(t)\, dt = 0, \ \forall i \qquad \int_0^T e_N(t)\, \hat{x}_N^*(t)\, dt = 0$$
- The error energy is given by
$$\mathcal{E}_{e_N} = \mathcal{E}_x - \mathcal{E}_{\hat{x}_N} = \mathcal{E}_x - \int_0 |\hat{x}_N(t)|^2 dt = \mathcal{E}_x - \sum_{i=-N}^{N} |a_i|^2$$

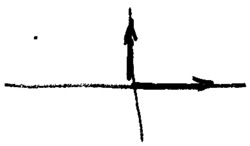
- Infinite basis sets. Some basis sets are drawn from a family with an infinite number of members, either countably infinite (e.g., F series) or uncountably infinite (e.g. F transform).

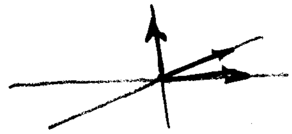
Can these basis sets represent <u>any</u> function in $L_2(T)$ or $L_2(\infty)$?

If $\mathcal{E}_{e_N} \to 0$ as $N \to \infty$ for any square integrable function $x(t)$, then the basis set is <u>complete</u>.

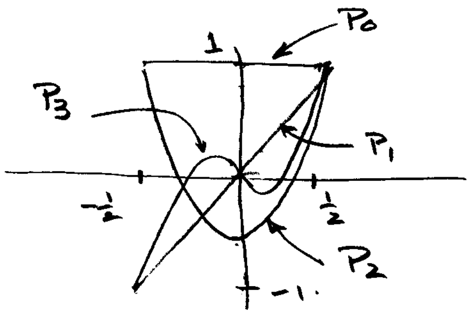
There is no square integrable and non zero function that is orthogonal to all the $\{v_i(t)\}$.

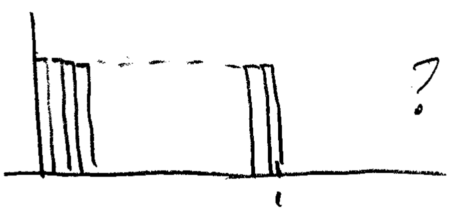
- The Fourier basis is complete, but it's not the only one (think of rotations, reflections ...)





Others include monomials (not orthonorm), Legendre polynomials, Chebyshev polynomials, Walsh functions ...





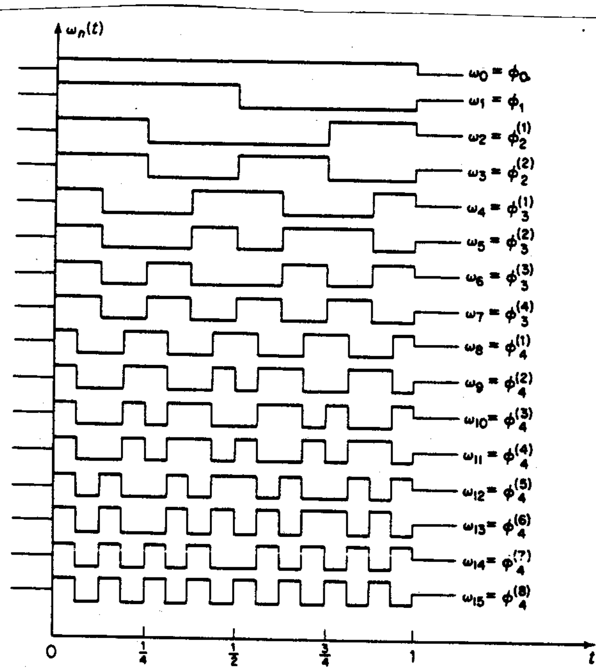
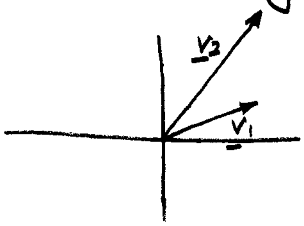


Figure 3.5. Walsh functions indexed according to number of sign changes in $0 < t < 1$.

- An orthonormal basis is clearly much easier to work with than nonorthogonal. So how to we convert an arbitrary basis to an orthonormal basis?

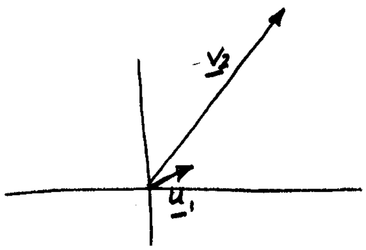
- The Gram-Schmidt orthogonalization procedure is one way to convert an arbitrary basis to an orthog one.

2D



construct an orthonormal basis from these.

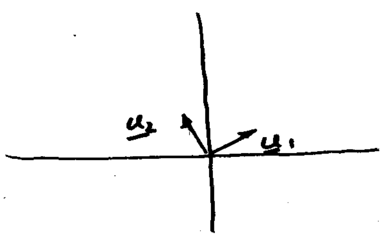
Step 1: choose either vector and normalize it to make the first unit vector $\underline{u}_1 = \frac{\underline{v}_1}{|\underline{v}_1|}$ (choose $\underline{v}_1$ arbitrarily)



Step 2: project the next $\underline{v}$ onto the unit vector basis (1-D so far) and calculate the error

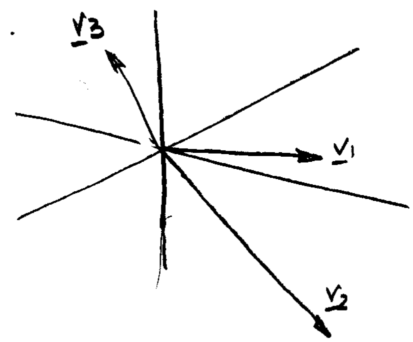
$$\underline{e}_2 = \underline{v}_2 - (\underline{v}_2, \underline{u}_1)\,\underline{u}_1$$

Step 3: Normalize the error to form the second orthonormal vector: $\underline{u}_2 = \frac{\underline{e}_2}{|\underline{e}_2|}$

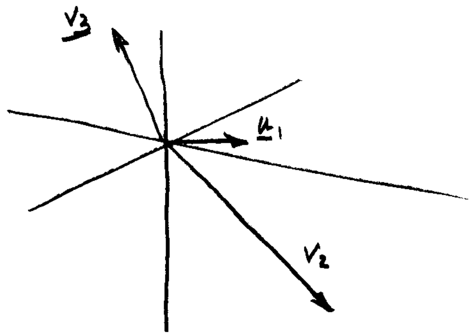


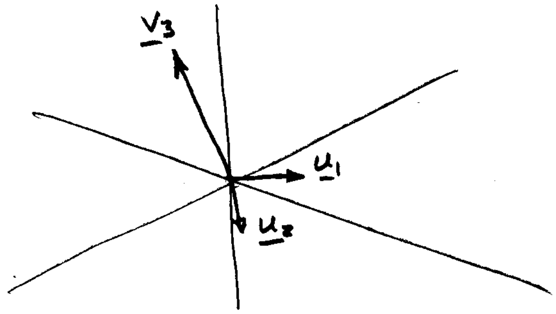
5.2.10

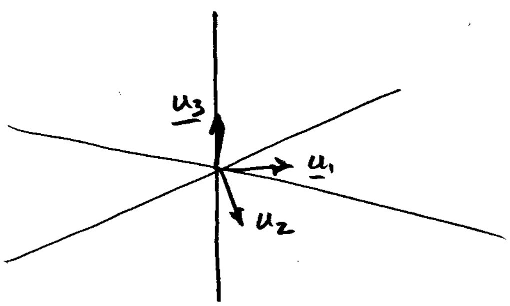
3D G-S



for convenience in sketching, $\underline{v}_1$ and $\underline{v}_2$ are in the plane, $\underline{v}_3$ is sticking up a little.







Pythagoras?

In general:

$\underline{u}_1 \leftarrow \underline{v}_1 / |\underline{v}_1|$

for $i \leftarrow 2$ to N,

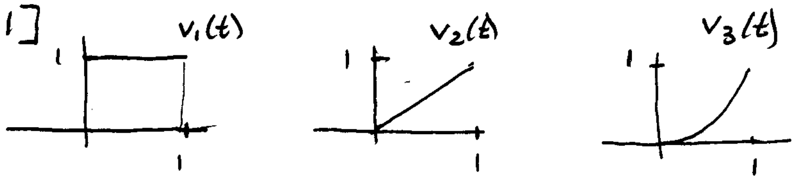
$$\underline{e} \leftarrow \underline{v}_i - \sum_{k=1}^{i-1} (\underline{v}_i, \underline{u}_k)\, \underline{u}_k$$

$$\underline{u}_i \leftarrow \underline{e} / |\underline{e}|$$

Depending on the order of selection, we can construct N! different bases.

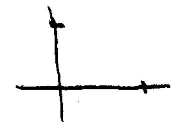
We can follow the G-S method for N indefinitely large, and whenever a new and LI vector $\underline{v}$ shows up,

<u>example</u> orthonormalize the monomials $1, t, t^2$ over $[0,1]$

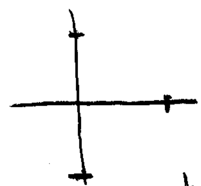


$i=1$: $u_1(t) = v_1(t)$; unit energy — easy.

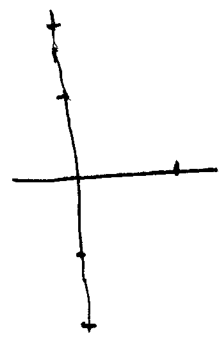
$i=2$: $\int_0^1 v_2(t)\,u_1(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}$, $\hat{v}_2(t) = \frac{1}{2}$



$e_2(t) = v_2(t) - \hat{v}_2(t) = t - \frac{1}{2}$



$\mathcal{E}_{e_2} = \int_0^1 (t-\frac{1}{2})^2 dt = \frac{1}{12}$



$u_2(t) = \underbrace{2\sqrt{3}}_{3.464}\,(t-\frac{1}{2})$

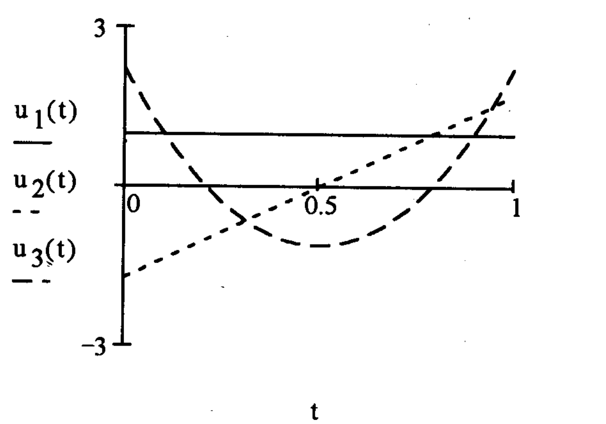
$i=3$: $\int_0^1 v_3(t)\,u_1(t)\,dt = \int_0^1 t^2 dt = \frac{1}{3}$

$\int_0^1 v_3(t)\,u_2(t)\,dt = \int_0^1 t^2\, 2\sqrt{3}\,(t-\frac{1}{2})\,dt = \frac{\sqrt{3}}{6}$

$\hat{v}_3(t) = \frac{1}{3} + \frac{\sqrt{3}}{6} \cdot 2\sqrt{3}\,(t-\frac{1}{2}) = t - \frac{1}{6}$

$e_3(t) = t^2 - t + \frac{1}{6}$ $\quad \mathcal{E}_{e_3} = \frac{1}{180}$ $\quad \sqrt{\mathcal{E}_{e_3}} = \frac{\sqrt{5}}{30}$

$u_3(t) = \frac{30}{\sqrt{5}}\,(t^2 - t + \frac{1}{6})$



<u>example</u>: Three zero-mean rvs v_1, v_2, v_3 have covariance matrix

$$R = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{12}^2 & \sigma_2^2 & \sigma_{23}^2 \\ \sigma_{13}^2 & \sigma_{32}^2 & \sigma_3^2 \end{bmatrix}$$

By G.S, obtain a set of uncorrelated unit variance rvs that span the same space.

<u>stage 1</u> $\quad u_1 = v_1/\sigma_1 \quad$ normalization

<u>stage 2</u> $\quad \hat{v}_2 = \underbrace{(v_2, u_1)}_{\sigma^2_{v_2 u_1}} u_1 = \frac{\sigma_{21}^2}{\sigma_1} \cdot u_1 = \frac{\sigma_{21}^2}{\sigma_1^2} v_1$

$$e_2 = v_2 - \hat{v}_2 = v_2 - \frac{\sigma_{21}^2}{\sigma_1^2} v_1$$

$$\sigma_{e_2}^2 = \overline{\left(v_2 - \frac{\sigma_{21}^2}{\sigma_1^2} v_1\right)^2} = \sigma_2^2 - 2\frac{\sigma_{21}^4}{\sigma_1^2} + \frac{\sigma_{21}^4}{\sigma_1^4}\sigma_1^2 = \sigma_2^2 - \frac{\sigma_{21}^4}{\sigma_1^2}$$

$$u_2 = \frac{e_2}{\sigma_{e_2}} = \frac{v_2 - \frac{\sigma_{21}^2}{\sigma_1^2} v_1}{\sqrt{\sigma_2^2 - \frac{\sigma_{21}^4}{\sigma_1^2}}}$$

Stage 3 $\quad \hat{v}_3 = (v_3, u_1) u_1 + (v_3, u_2) u_2$

$$= \frac{\sigma_{31}^2}{\sigma_1} \frac{v_1}{\sigma_1} + \left(\frac{\sigma_{32}^2 - \frac{\sigma_{21}^2 \sigma_{31}^2}{\sigma_1^2}}{\sqrt{\sigma_2^2 - \frac{\sigma_{21}^4}{\sigma_1^2}}} \right) \frac{v_2 - \frac{\sigma_{21}^2}{\sigma_1^2} v_1}{\sqrt{\sigma_2^2 - \frac{\sigma_{21}^4}{\sigma_1^2}}}$$

simplify and carry on...