4.2 Rayleigh and Rice fading

Rayleigh Fading

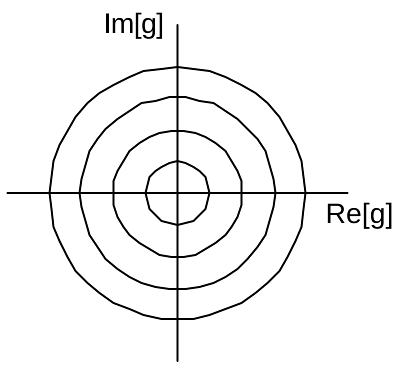
From this point on, we assume that the channel complex gain or transfer function is Gaussian. For flat fading with no LOS component (i.e., zero mean), we have the variance

$$\sigma_g^2 = \frac{1}{2}\cdot E\left[\left(|g(t)|\right)^2\right] = \frac{1}{2}\cdot\left(E\left(g_I(t)^2\right) + E\left(g_Q(t)^2\right)\right) \tag{4.2.1}$$

Note that the real and imaginary components are individually Gaussian with variance σ_g^2. The probability density function is

$$p_g(g) = \frac{1}{2\cdot\pi\cdot\sigma_g^2}\cdot\exp\left[-\frac{1}{2}\cdot\frac{\left(|g|\right)^2}{\sigma_g^2}\right] \tag{4.2.2}$$

and its isoprobability contours are circles centred on the origin:



If we change to polar coordinates $g = g_I + j\cdot g_Q = r\cdot e^{j\cdot\theta}$ then standard transformations [**Papo84, Proa95, Lee82**] give the joint pdf as

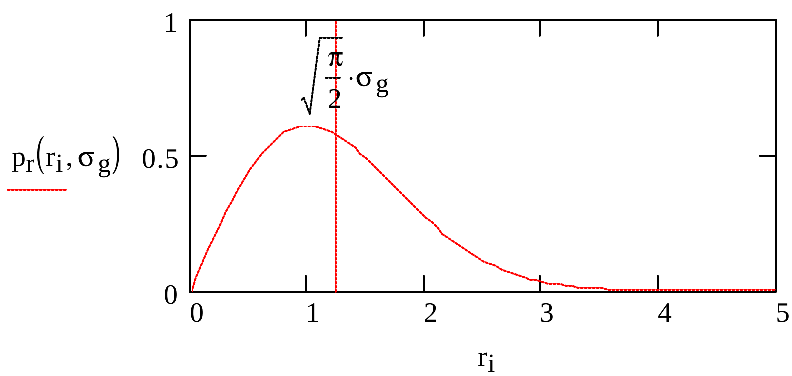
$$p_{r\theta}(r,\theta) = \frac{r}{2\cdot\pi\cdot\sigma_g^{\,2}}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma_g^{\,2}}\right) \qquad (4.2.3)$$

Clearly, r and θ are independent, since the joint pdf is the product of their individual pdfs, given by

$$p_r(r,\sigma_g) := \frac{r}{\sigma_g^{\,2}}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma_g^{\,2}}\right) \quad r \ge 0 \quad \text{and} \quad p_\theta(\theta) = \frac{1}{2\cdot\pi} \quad -\pi \le \theta < \pi \quad (4.2.4)$$

This pdf of the amplitude r is the *Rayleigh distribution*, and this type of fading (no LOS component) is termed Rayleigh fading. Let's see what it looks like.

$i := 0..100 \qquad r_i := \frac{5}{100}\cdot i$ Your choice of σ_g: $\sigma_g := 1$



Rayleigh distribution of amplitude

mean	mode	standard deviation
$\mu_r = \sqrt{\frac{\pi}{2}}\cdot\sigma_g$	σ_g	$\sigma_r = \sqrt{2-\frac{\pi}{2}}\cdot\sigma_g$

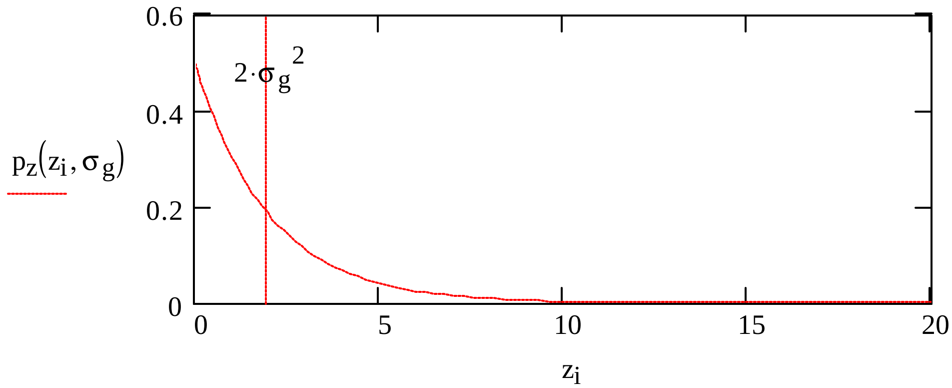
It's often easier to work with the squared amplitude (twice as large as instantaneous power)

$$z = r^2 = (|g|)^2$$

because it is exponentially distributed. This follows from a simple change of variables to *z* from *r* in the Rayleigh pdf. Alternatively, note that $z = g_I^2 + g_Q^2$, and that the sum of independent squared Gaussian variates has the χ^2 distribution, and that the χ^2 distribution with two degrees of freedom is exponential. In any case, the pdf of *z* is

$$p_z(z, \sigma_g) := \frac{1}{2 \cdot \sigma_g^2} \cdot e^{-\frac{z}{2 \cdot \sigma_g^2}} \qquad z \geq 0 \qquad (4.2.5)$$

Again, plot it: $z_i := (r_i)^2$ Your choice of σ_g: $\sigma_g := 1$



$\sigma_g^2 = 1$

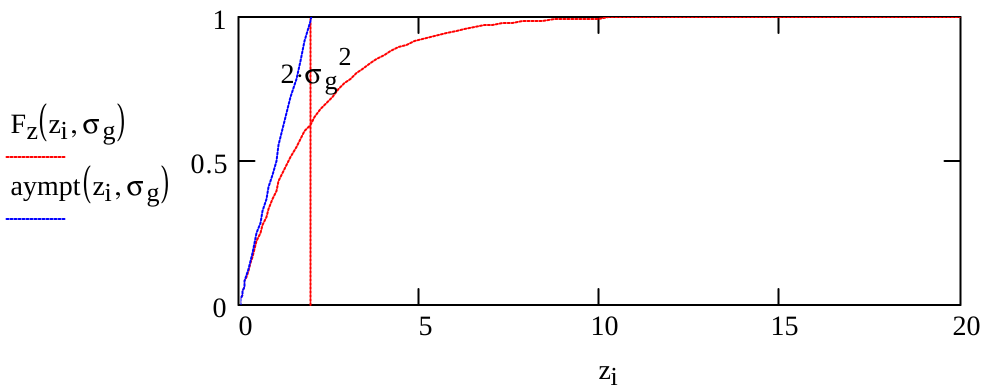
Exponentially distributed squared magnitude

The mean equals the decay constant and so does the standard deviation.

$$\mu_z = 2 \cdot \sigma_g^2 \qquad \sigma_z = 2 \cdot \sigma_g^2$$

The cumulative distribution function of *z* and its asymptote are

$$F_z(z,\sigma_g) := 1 - \exp\left(-\frac{z}{2\cdot\sigma_g^{\,2}}\right) \qquad \text{aympt}(z,\sigma_g) := \frac{z}{2\cdot\sigma_g^{\,2}} \tag{4.2.6}$$



This is for

$\sigma_g^{\,2} = 1$

The asymptote gives us two very useful rules of thumb. Remember them:

* **The probability that received power is 10 dB or more below the mean level (a 10 dB fade) is 10%; probability of a 20 dB fade is 1%; probability of a 30 dB fade is 0.1%; etc.** Now go back to the fade graph in **Section 3.1** and see whether this seems to be true (recalculate a few times). Remember that with λ/50 sampling, some deep fades may be missed, so you are really looking at the fraction of the number of samples *M* that falls below the threshold.

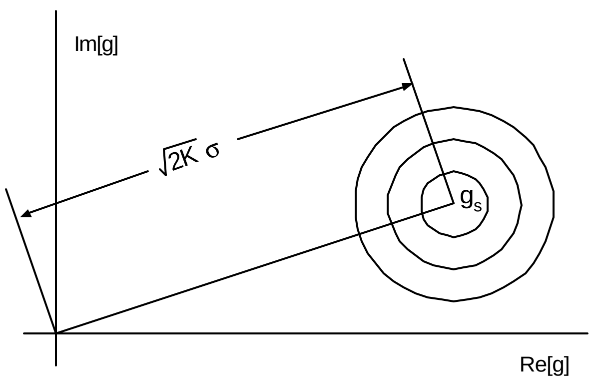
* **The probability that the power drops below a given level decreases only inversely with increasing average power** $s_g^{\,2}$**.** That's important - and disappointing - if the level is a threshold below which operation is unacceptable, since doubling the average power only cuts the probability in half!

Rice Fading

In mobile satellite systems, or in land mobile radio in suburban and rural areas, the signal is often received with a LOS component which produces Rice fading. The total gain

$$g = g_s + g_d \tag{4.2.7}$$

is the sum of a constant specular (or LOS or discrete) component g_s and a zero mean Gaussian diffuse (or scattered) component g_d, so that g is a nonzero mean Gaussian variate. The specular component has K times the power of the diffuse component (the Rice K-factor), so that K=0 gives Rayleigh fading and K==>∞ gives a constant channel. But be careful - some literature (mostly in the mobile satellite area) uses K as the ratio of diffuse to specular power, the reciprocal of the conventional definition. The sketch shows the isoprobability contours.



Denote the variance of the diffuse component by σ^2. From the power ratio we have the magnitude of the specular component.

$$\frac{1}{2}\cdot E\left[\left(\left|g_d\right|\right)^2\right] = \sigma^2 \qquad \left|g_s\right| = \sqrt{2\cdot K}\cdot\sigma \tag{4.2.8}$$

The total average power in g is then

$$\frac{1}{2}\cdot E\left[\left(\left|g\right|\right)^2\right] = \frac{1}{2}\cdot E\left[\left(\left|g_s\right|\right)^2\right] + \frac{1}{2}\cdot E\left[\left(\left|g_d\right|\right)^2\right] = K\cdot\sigma^2 + \sigma^2 = \sigma^2\cdot(1+K) \qquad (4.2.9)$$

and the mean and variance of *g* are $\mu_g = g_s$ and $\sigma_g^{\ 2} = \sigma^2$. Its pdf is Gaussian:

$$p_g(g) = \frac{1}{2\cdot\pi\cdot\sigma_g^{\ 2}}\cdot\exp\left[-\frac{1}{2}\cdot\frac{\left(\left|g-\mu_g\right|\right)^2}{\sigma_g^{\ 2}}\right] \qquad (4.2.10)$$

Changing to polar coordinates makes $z=r^2$ non-central χ^2 with mean $\sigma^2(1+K)$ and two degrees of freedom. Alternatively, the pdf of r is Rician:

$$p_{r\_K}(r,K,\sigma) := \frac{r}{\sigma^2}\cdot\exp\left(-\frac{r^2}{2\cdot\sigma^2} - K\right)\cdot I0\left(\frac{r\cdot\sqrt{2\cdot K}}{\sigma}\right) \qquad (4.2.11)$$

From the isoprobability sketch above, it is clear that the phase angle is not independent of the amplitude. The unconditional pdf of the phase angle for a real specular component (i.e., zero mean phase angle) is obtained by adapting [**Proa89**, eqn. 4.2.103]. First, the *Q* function:

$$Q(x) = \frac{1}{\sqrt{2\cdot\pi}}\cdot\int_x^{\infty} e^{-\frac{\alpha^2}{2}}\,d\alpha \qquad \text{or} \qquad Q(x) := \text{cnorm}(-x)$$

$$p_{\theta\_K}(\theta,K,\sigma) := \frac{1}{2\cdot\pi}\cdot e^{-K}\cdot\left[1+\sqrt{4\cdot\pi\cdot K}\cdot\cos(\theta)\cdot e^{K\cdot\cos(\theta)^2}\cdot\left(1-Q\left(\sqrt{2\cdot K}\cdot\cos(\theta)\right)\right)\right]$$

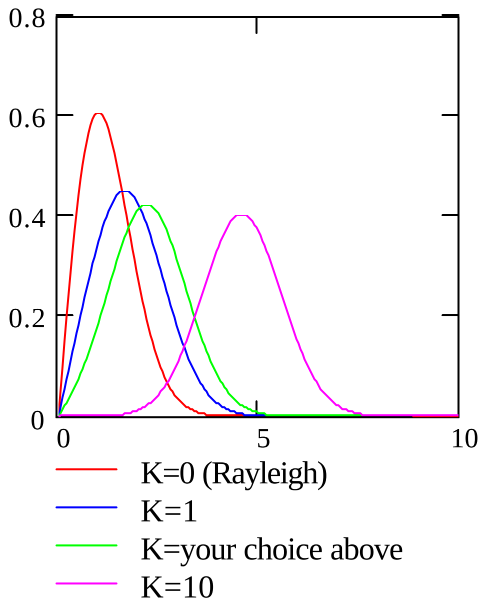
(4.2.12)

Now let's see what these pdfs look like.

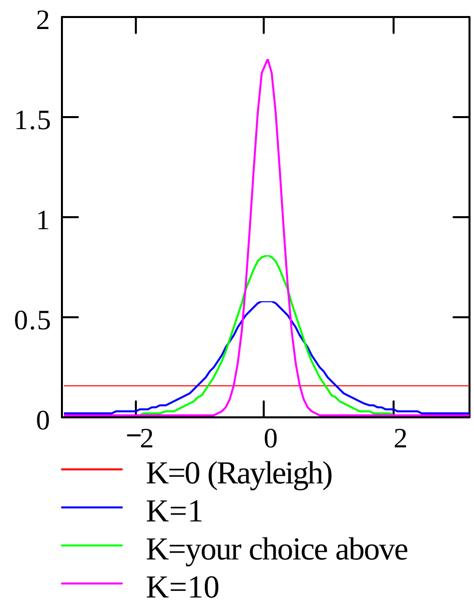
plot ranges: $r := 0, 0.05 .. 10$ $\theta := -\pi, -0.98 \cdot \pi .. \pi$

Your choice of *K* and σ below

$K := 2$ $\sigma := 1$



Rice amplitude pdf



Rice phase pdf (for zero mean)

Inspection of the graphs suggests that they can be approximated by Gaussian pdfs for large *K*. It's easy to see why if you rotate the coordinates for g_d to resolve it into a radial component (along the same line as g_s) and a transverse component (orthogonal to the radial component). For large *K*, the transverse component makes little difference to the amplitude, which is then well modeled by the Gaussian radial component with the specular component as a mean. Similarly, the radial component makes little difference to the phase, which is then well modeled by the Gaussian transverse component divided by the specular amplitude. Therefore,

* approx amplitude pdf, large *K*: Gaussian, mean $\sqrt{2\cdot K}\cdot\sigma$ and standard deviation σ

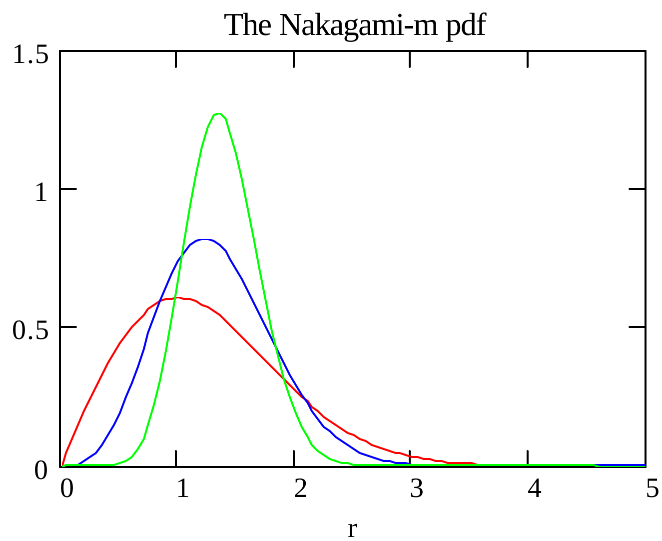
* approx phase pdf, large *K*: Gaussian, mean $\arg(g_s)$ and standard deviation $\frac{1}{\sqrt{2\cdot K}}$

Nakagami Density

When you did the **experiment in Appendix J**, you noticed that the Rayleigh pdf was a fairly rough fit to the histogram of experimental values if the number of paths was small. Many authors report that a better approximation of their experimental measurements is obtained with the Nakagami-m distribution, given by [**Naka60**]

$$p_{r\_N}(r,m,\sigma) := \frac{2}{\Gamma(m)}\cdot\left(\frac{m}{2\cdot\sigma^2}\right)^m\cdot r^{2\cdot m-1}\cdot e^{\frac{-m\cdot r^2}{2\cdot\sigma^2}} \quad \text{for} \quad r \geq 0 \text{ and } \quad m \geq \frac{1}{2} \qquad (4.2.13)$$

where m is the order of the pdf, $2\sigma^2$ is the mean square value and $\Gamma(m)$ is the gamma function (equal to $(m-1)!$ for integers). For m=1, Nakagami reduces to Rayleigh. Let's have a look at it for $\sigma := 1$:



red: m=1 (equals Rayleigh)

blue: m=2

green: m=3

You can see that increasing the order *m* of the Nakagami distribution changes its character from that of purely scattered fading to fading with a LOS component. For modeling these channels, it is therefore a reasonable alternative to the Rice pdf which it resembles. For larger values of *m*, just as for larger values of *K* in the Rice pdf, it can be approximated in turn by a Gaussian pdf.

Why bother with this new pdf, when we already have the Rice pdf? One reason is its simplicity. For example, by change of variables, the squared amplitude $z=r^2$ has a gamma pdf:

$$p_{z\_N}(z,m,\sigma) := \frac{1}{\Gamma(m)} \cdot \left(\frac{m}{2\cdot\sigma^2}\right)^m \cdot z^{m-1} \cdot e^{\frac{-m\cdot z}{2\cdot\sigma^2}} \qquad \text{for} \quad z \geq 0 \text{ and } \quad m \geq \frac{1}{2} \qquad (4.2.14)$$

This looks more complicated than it is - just focus on the variation with *z* and it looks like functions you have seen before in your undergraduate course on linear systems and Laplace transforms. Consequently, its characteristic function (Laplace transform of pdf) is

$$M_{z\_N}(s,m,\sigma) := \left(\frac{m}{2\cdot\sigma^2} \cdot \frac{1}{s + \frac{m}{2\cdot\sigma^2}}\right)^m \qquad (4.2.15)$$

which has an *m*th order pole at $s = \frac{-m}{2\cdot\sigma^2}$ You can obtain many analytical results conveniently with these expressions, in contrast to the Rice pdf (4.2.11), with its embedded Bessel function. For representative work using the Nakagami pdf see [**Pate97**], [**Ugwe97**].