APPENDIX H
MAXIMAL RATIO COMBINING

- Problem: We have M complex random variables x_m, $m=1,..M$, each with a mean value μ_m (the signal) and variance σ_m^2 (the noise variance).

  Form a linear combination to improve signal

$$\hat{x} = \sum_{m=1}^{M} w_m^* x_m$$

  What choice of weights maximizes the SNR

$$\frac{S}{N} = \frac{\left|\sum_m w_m^* \mu_m\right|^2}{\sum_m |w_m|^2 \sigma_m^2} \quad ?$$

- Solution will be unique only to within a scale factor.

- Could go through Schwartz inequality, or could treat as a Rayleigh quotient, but it's more fun to do a constrained maximization.

$$\max \left|\sum_m w_m^* \mu_m\right|^2 \quad \text{with} \quad \sum_m |w_m|^2 \sigma_m^2 = 1$$

- With Lagrange multiplier, maximize

$$J = \underline{w}^{+}\underline{\mu}\,\underline{\mu}^{+}\underline{w} - \lambda\,\underline{w}^{+}\Sigma\Sigma\underline{w}\ ,\qquad \Sigma = \mathrm{diag}(\sigma_1, \cdots \sigma_M)$$

- Take gradient (see Appendix K)

$$\nabla_{\underline{w}} J = 2\underline{\mu}\,\underline{\mu}^{+}\underline{w} - \lambda\Sigma\Sigma\underline{w} = 0$$

Define $\underline{u} = \Sigma\underline{w} \qquad \underline{w} = \Sigma^{-1}\underline{u}$

and premult by Σ^{-1}:

$$\Sigma^{-1}\underline{\mu}\,\underline{\mu}^{+}\Sigma^{-1}\underline{u} = \lambda\underline{u}\ ,\qquad \Sigma^{-1}\underline{\mu} = \underline{b}$$

$$\underline{b}\,\underline{b}^{+}\underline{u} = \lambda\underline{u}$$

Solved by $\quad \underline{u} = \alpha\underline{b}\ ,\quad \lambda = \underline{b}^{+}\underline{b}\ ,\quad \alpha$ arbitrary

$$\underline{w} = \Sigma^{-1}\underline{u} = \alpha\Sigma^{-1}\Sigma^{-1}\underline{\mu} = \alpha\Sigma^{-2}\underline{\mu}$$

so

$$\boxed{w_m = \alpha\frac{\mu_m}{\sigma_m^2}} \qquad \boxed{\left(\frac{S}{N}\right)_{max} = \underline{\mu}^{+}\Sigma^{-2}\underline{\mu} = \sum_{m=1}^{M}\frac{|\mu_m|^2}{\sigma_m^2}}$$

- The conjugation in the linear combination

$$\hat{x} = \sum_{m=1}^{M}\frac{\mu_m^*}{\sigma_m^2}x_m$$

causes the signal components to be phase-aligned and amplitude-weighted