Appendix K: Derivatives and Gradients WRT Complex Variables

- Suppose $z = x + jy$ is a variable

$$f(z) = u(x,y) + j\,v(x,y) \text{ is a complex f'n of } z$$

- If $f(z)$ is analytic, it satisfies the Cauchy Riemann conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

and there is no trouble defining a derivative

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial u}{\partial jy} + j\frac{\partial v}{\partial jy} = j\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z)}{\Delta z} \text{ independent of direction } \measuredangle \Delta z.$$

- What if $f(z)$ is <u>not</u> analytic? Such as $f(z) = |z|^2$!

How can we define a derivative?

Here's a good way:

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - j\frac{\partial f}{\partial y}\right)$$

It conforms with the usual definition for analytic functions

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial jy}\right) = \frac{\partial f}{\partial x} + j\frac{\partial f}{\partial x} = \frac{\partial u}{\partial jy} + j\frac{\partial v}{\partial jy}$$

and for (e.g.) $f(z) = |z|^2$, $v(z) \equiv 0$, a real function

$$\frac{df}{dz} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - j\frac{\partial u}{\partial y}\right)$$

- Try it on elementary functions

$f(z) = 1$: $\quad \frac{df}{dz} = 0$

$f(z) = z$: $\quad \frac{dz}{dz} = \frac{1}{2}\left(\frac{\partial z}{\partial x} - j\frac{\partial z}{\partial y}\right) = \frac{1}{2}\left(1 - j(j)\right) = 1$

$f(z) = z^*$: $\quad \frac{dz^*}{dz} = \frac{1}{2}\left(\frac{\partial z^*}{\partial x} - j\frac{\partial z^*}{\partial y}\right) = \frac{1}{2}\left(1 - j(-j)\right) = 0$

$f(z) = z^2 = z\,z$: $\frac{dz^2}{dz} = z\frac{dz}{dz} + \frac{dz}{dz}z = 2z$

$f(z) = |z|^2 = z\,z^*$: $\frac{d|z|^2}{dz} = z\frac{dz^*}{dz} + \frac{dz}{dz}z^* = z^*$

- Next, extend this to a gradient. As usual, it is the derivative of a scalar wrt a vector

$$\underline{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \qquad \Phi(z) \text{ is scalar function of vector } \underline{z}$$

$$\nabla_{\underline{z}}\Phi = \begin{bmatrix} \frac{\partial\Phi}{\partial z_1} \\ \vdots \\ \frac{\partial\Phi}{\partial z_N} \end{bmatrix}^*$$

The conjugate is for historical reasons. A simplifying convention.

Examples:

→ $\Phi(\underline{z}) = \underline{a}^+\underline{z} = \sum_{i=1}^{N} a_i^* z_i$

$$\nabla_z\Phi = \begin{bmatrix} a_1^* \\ \vdots \\ a_N^* \end{bmatrix}^* = \underline{a}$$

→ $\Phi(z) = \underline{z}^+\underline{a} = \sum_{i=1}^{N} z_i^* a_i$, $\nabla_z\Phi = \underline{0}$

→ $\Phi(z) = \underline{z}^+A\,\underline{z} = \underline{z}^+(A\underline{z}) = (\underline{z}^+A)\underline{z}$

$$\nabla_{\underline{z}}\Phi(z) = 0 + (z^+A)^+ = A^+\underline{z}$$