2.2 Second Order Statistics of Complex Signals

- In this section, we look at 2^{nd} order stats — power, autocorrelation, power spectrum — to see how real bandpass and complex envelope stats are related.

- First, energy. The energy of $\tilde{v}(t)$ is half that of its complex envelope $v(t)$:

$$\int \tilde{v}^2(t)\, dt = \frac{1}{2} \int |v(t)|^2 dt$$

$\boxed{\text{Why?}}$

or by Parseval

$$\int |\tilde{V}(f)|^2 df = \frac{1}{2} \int |V(f)|^2 df$$

Because the former is the actual transmitted energy, it is common in communications to write

$$E_v = \frac{1}{2} \int |v(t)|^2 dt = \frac{1}{2} \int |V(f)|^2 df$$

- Next, random processes. Define correlation functions

$$R_x(\tau) = \frac{1}{2} E\left[x(t)\, x^*(t-\tau)\right]$$

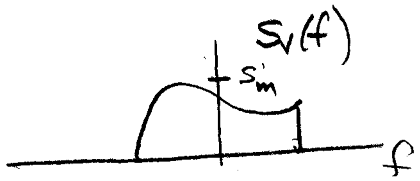
$$R_{xy}(\tau) = \frac{1}{2} E\left[x(t)\, y^*(t-\tau)\right]$$

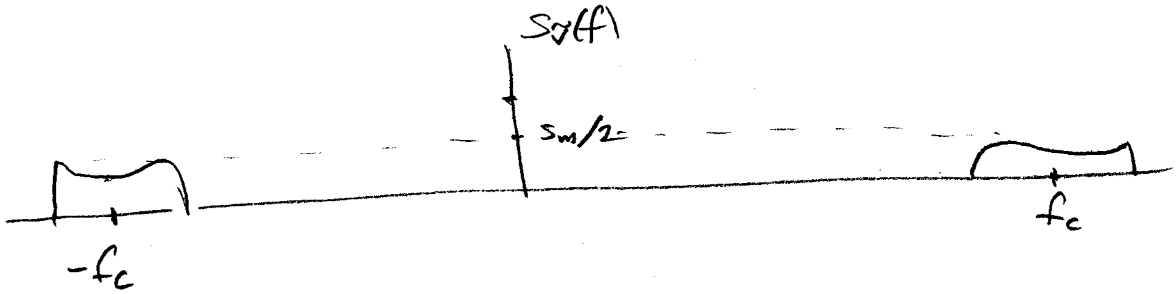
$$P_x = R_x(0)$$

- Statistics of the corresponding bp process are given by modulation relationships like those linking $v(t)$ and $\tilde{v}(t)$:

$$R_{\tilde v}(\tau) = E\left[\tilde v(t)\tilde v(t-\tau)\right] = \mathrm{Re}\left[R_v(\tau)e^{j2\pi f_c \tau}\right]$$

$$S_{\tilde v}(f) = \mathcal{F}\left[R_{\tilde v}(\tau)\right] = \frac{1}{2}S_v(f) + \frac{1}{2}S_v(-(f+f_c))$$





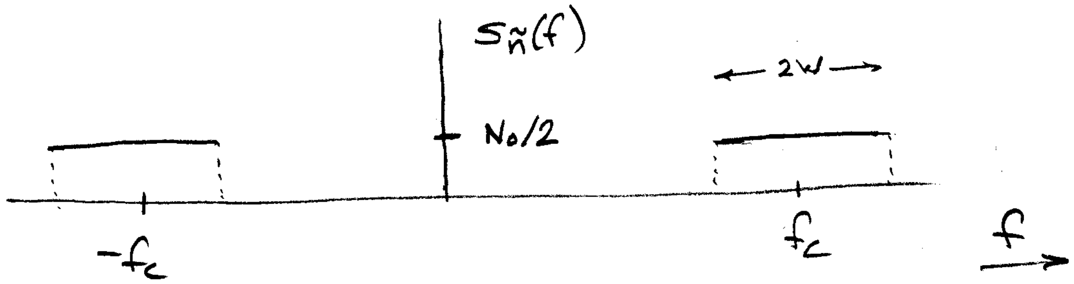
- $S_v(f)$ is real $\boxed{\text{Why?}}$

but not necessarily symmetric. $\boxed{\text{Why not?}}$

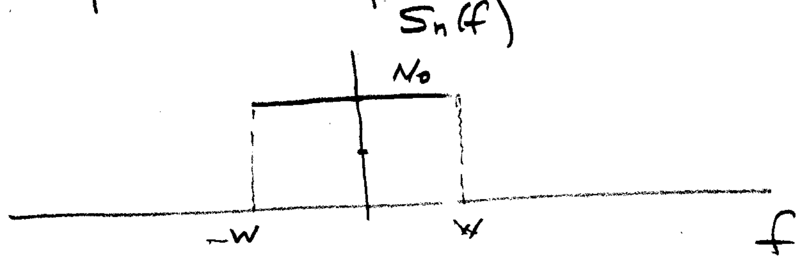
Symmetric if and only if $v_r(t)$ and $v_i(t)$ are uncorrelated. $\boxed{\text{Why?}}$

- White noise, in particular...

The real bandpass signal $\tilde{n}(t)$ has PSD



and complex envelope PSD



Are the powers the same?

- More detail Appendix B2

- Projection of white noise onto set of functions $u_1(t), \ldots, u_N(t)$, <u>not</u> necessarily orthonormal.

$n(t)$ complex envelope, $R_n(\tau) = N_0\, \delta(\tau)$

$$n_i = \int n(t)\, u_i^*(t)\, dt \qquad \text{zero mean noise}$$

so

$$\sigma_i^2 = \tfrac{1}{2} E\left[|n_i|^2\right] = \tfrac{1}{2} E \iint n(t)\, n^*(s)\, u_i^*(t)\, u_i(s)\, dt\, ds$$

$$= N_0 \iint \delta(t-s)\, u_i^*(t)\, u_i(s)\, dt\, ds$$

$$= N_0 \int |u_i(t)|^2 dt$$

and covariance

$$\sigma_{ik}^2 = \tfrac{1}{2} E\left[n_i\, n_k^*\right] = \tfrac{1}{2} E \iint n(t)\, n^*(s)\, u_i^*(t)\, u_k^*(s)\, dt\, ds$$

$$= N_0 \int u_i(t)\, u_k^*(t)\, dt$$

- In particular, noise components wrt orthogonal functions are uncorrelated.

- As a vector

$$\underline{n} = \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix} = \int n(t)\, \underline{u}^*(t)\, dt = \int n(t) \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix}^* dt$$

Corr'n matrix

$$R_n = \tfrac{1}{2} E\left[\underline{n}\, \underline{n}^+\right] = \tfrac{1}{2} \iint \overline{n(t)\, n^*(s)}\, \underline{u}^*(t)\, \underline{u}^T(s)\, dt\, ds$$

$$= N_0 \underbrace{\int \underline{u}^*(t)\, \underline{u}^T(t)\, dt}_{\Phi_u} = N_0\, \Phi_u$$

- Projection of white noise onto subspace spanned by $u_1(t), \dots u_N(t)$:

$$\hat{n}(t) = \sum_{i=1}^{N} \nu_i\, u_i(t) \quad = \underline{\nu}^T \underline{u}(t)$$

where coefficients $\{\nu_i\}$ are selected to minimise

$$\mathcal{E}_\perp = \int |n(t) - \hat{n}(t)|^2 dt$$

Solution
$$\underline{\nu} = \Phi_u^{-1}\, \underline{n}$$

- The proof is worth following through, since it illustrates classic linear approximation, but in complex formulation.

- Expand energy in the error *

$$\mathcal{E}_\perp = \int |n(t) - \sum_{i=1}^{N} \nu_i u_i(t)|^2 dt \quad = \int |n(t) - \underline{\nu}^T \underline{u}(t)|^2 dt$$

$$= \mathcal{E}_n - \underline{\nu}^+ \int n(t)\, \underline{u}^*(t)\, dt \; - \int \underline{u}^T(t)\, n^*(t)\, dt \cdot \underline{\nu}$$
$$+ \; \underline{\nu}^+ \int \underline{u}^*(t)\, \underline{u}^T(t)\, dt \; \underline{\nu}$$

$$= \mathcal{E}_n - \underline{\nu}^+ \underline{n} - \underline{n}^+ \underline{\nu} + \underline{\nu}^+ \Phi_u \underline{\nu}$$

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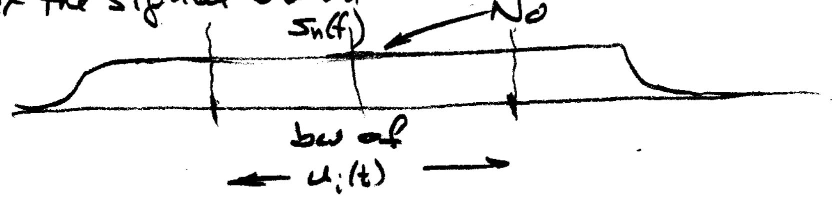
* But white noise has infinite power, so $\mathcal{E}_\perp$, $\mathcal{E}_n$ are infinite ?!?
Solution — dodge the issue and make $S_n(f)$ flat only over the signal band



— Minimise $\mathcal{E}_\perp$ wrt $\underline{v}$; it's a scalar, so set the gradient $\nabla_{\underline{v}} \mathcal{E}_\perp = \underline{0}$. Next problem: coefficients $\underline{v}$ are complex, but $\mathcal{E}_\perp$ is not analytic.

Solution 1: take grad wrt re and im parts separately

Solution 2: use the compact formalism of Appendix K.

$$\nabla_{\underline{v}} \mathcal{E}_\perp = \nabla_{\underline{v}} \mathcal{E}_n - \nabla_{\underline{v}} \underline{v}^\dagger \underline{n} - \nabla_{\underline{v}} \underline{n}^\dagger \underline{v} + \nabla_{\underline{v}} \underline{v}^\dagger \Phi_U \underline{v}$$

$$= \underline{0} - \underline{0} - \underline{n} + \Phi_U \underline{v} = \underline{0}$$

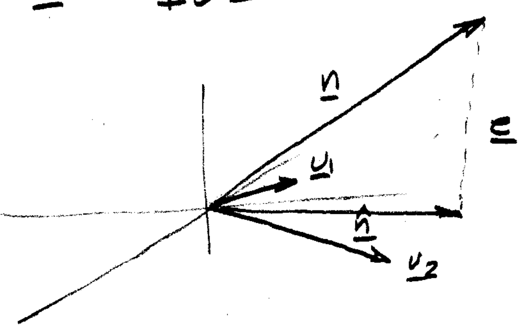
so $\boxed{\Phi_U \underline{v} = \underline{n}, \quad \underline{v} = \Phi_U^{-1} \underline{n}}$

These are the "normal equations": matrix of inner prods of basis functions (Gram matrix Φ_U) and vector of inner prods of function being approx'd and basis f'ns ($\underline{n}$).

— Important property: error $e(t) = n(t) - \hat{n}(t)$ is orthogonal to any vector in the subspace, including $\hat{n}(t)$. Easy proof: get vector of inner prods of $e(t)$ and basis functions

$$\underline{e} = \int e(t) \underline{u}^*(t)\, dt = \int n(t) \underline{u}^*(t)\, dt - \int \underline{u}^*(t)\, \underline{u}^T(t)\, \underline{v}\, dt$$

$$= \underline{n} - \Phi_U \underline{v} = \underline{0}$$



$\underline{e} \perp$ all vectors in the subspace.

- Summary:
  - → The coefficients $\underline{v}$ and inner prods $\underline{n}$ are equivalent, since $\Phi_U \underline{v} = \underline{n}$
  - → Either is sufficient to characterise the projection $\hat{n}(t)$ of $n(t)$ onto $\{v_i\}$ space
  - → That part of noise falling outside U space is orthog to everything in U
  - → Components of $n(t)$ on basis vectors orthog to U are uncorrelated with $\underline{n}$

- Finally, if vector is complex Gaussian, then

$$p_{\underline{n}}(\underline{n}) = \frac{1}{(2\pi)^N |R_n|} \exp\left(-\frac{1}{2}\, \underline{n}^{+} R_n^{-1} \underline{n}\right)$$

where $R_n = \frac{1}{2} E[\underline{n}\,\underline{n}^{+}]$

What is it for non-zero mean ?

with characteristic function

$$M_n(\underline{\omega}) = E\left[e^{j\underline{\omega}^{+}\underline{n}}\right]$$

$$= \exp\left(-j\,\frac{1}{2}\,\underline{\omega}^{+} R_n\, \underline{\omega}\right)$$

or with mean $\underline{\mu} = E[\underline{n}]$

$$M_n(\underline{\omega}) = \exp(j\underline{\mu}^{+}\underline{\omega})\, \exp\left(-j\frac{1}{2}\underline{\omega}^{+} R_n\, \underline{\omega}\right)$$