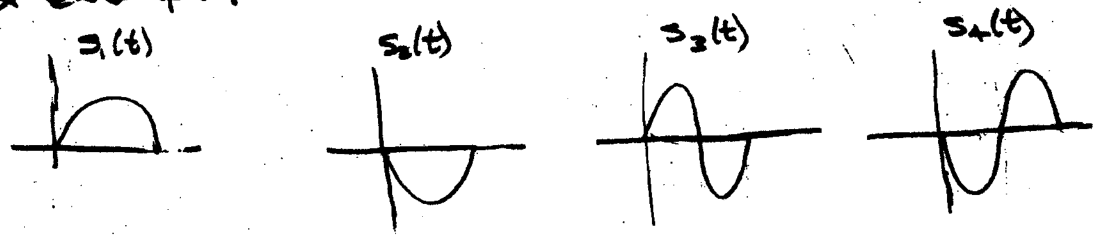
2.3 Detection in White Noise

Here we finally apply what we have learned so far to the basic problem of digital transmission – detection of isolated pulses in white noise.

> Also see your undergrad textbook or ENSC428 notes.

Transmitted and Received Waveforms

- This is what we send: one of M possible waveforms, for example:

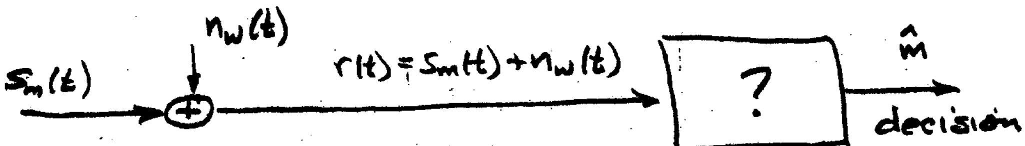


They have *a priori* probabilities $P(s_i)$ (*a priori* here means "before we look at the received signal")

Commonly: – M is a power of 2
– signals are equiprobable

but not always!

- This is what we receive:



white noise, PSD N_0, Gaussian

We make our decision about what signal was sent using the waveform $r(t)$.

AWGN =

Obtaining Sufficient Statistics

- Dealing with a complete waveform $r(t)$, $-\infty < t < \infty$ is awkward. We want to reduce it to a finite number of parameters without losing any information relevant to the decision. Here's how...

- Using <u>any</u> basis of the signal space, project $r(t)$ onto that basis. This represents the signal $s_m(t)$ exactly and the noise $n_w(t)$ approximately.

$$\hat{r}(t) = \hat{s}_m(t) + \hat{n}(t) = s_m(t) + \hat{n}(t)$$

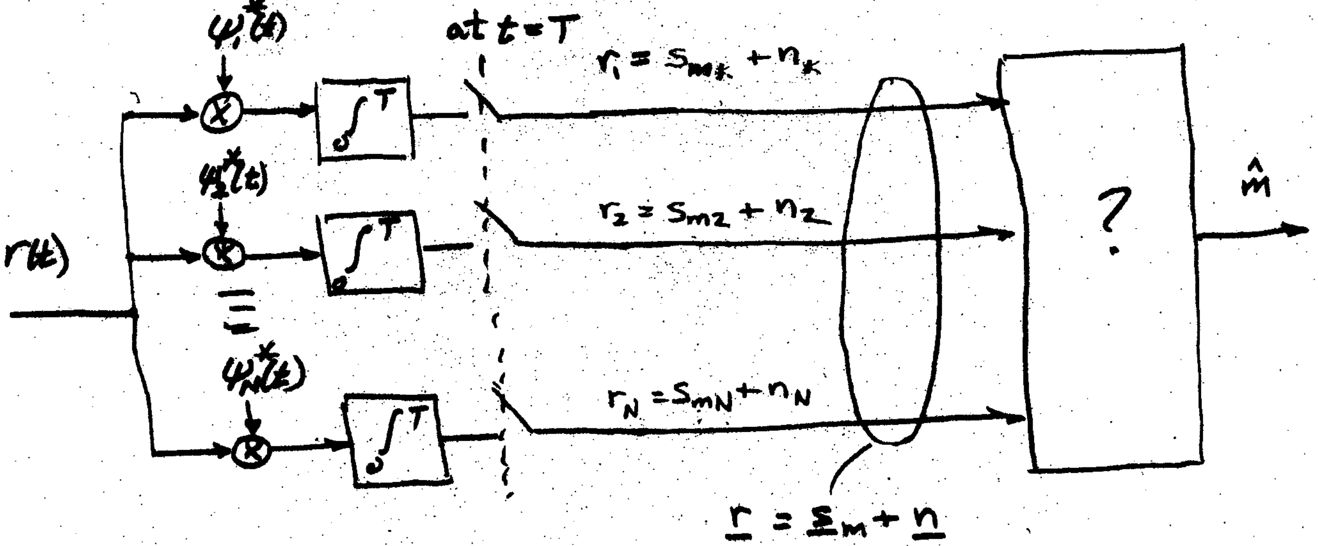
so $$r(t) = s_m(t) + \hat{n}(t) + e(t)$$

- "Theorem of Irrelevance"
  - Note $e(t)$ lies wholly in the orthogonal complement of the signal subspace. The basis functions of that complementary subspace are orthogonal to every function in the signal space, including $s_m(t)$, $\hat{n}(t)$.
  - Recall that white noise projections onto orthogonal functions are uncorrelated random variables, hence independent if Gaussian
  - Hence $e(t)$ gives no information about either $\hat{n}(t)$ or $s_m(t)$, so we don't need it.

    The projection $\hat{r}(t)$ is sufficient.

More on sufficient statistics: your undergrad stats text or Appendix F

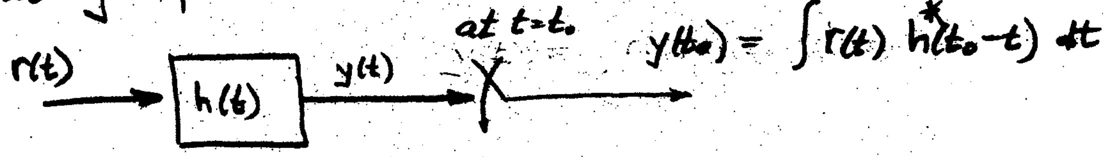
- Conceptually, it's simplest to use an orthonormal basis $\{\psi_i(t)\}$, $i=1..N$. Obtaining the set of sufficient statistics becomes straightforward:



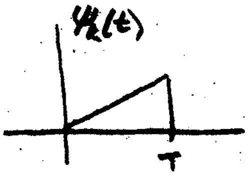
A set of correlators like this is the most common way to reduce an entire waveform to N numbers.

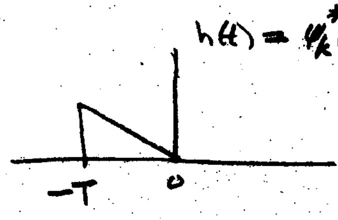
ASIDE

- An alternative implementation — the matched filter — is often more convenient for analysis and for analog implementations.



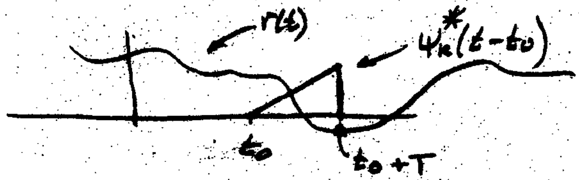
If we make the impulse response a "pre-flipped" version of $\psi_k^*(t)$, then we get a sliding correlation, or sliding inner product.

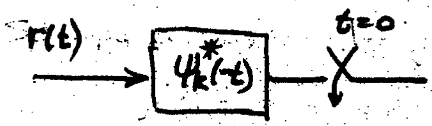




not causal yet, but we'll fix that

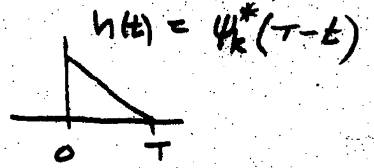
so $y(t_0) = \int r(t)\, h(t_0 - t)\, dt = \int r(t)\, \psi_k^*(t - t_0)\, dt$



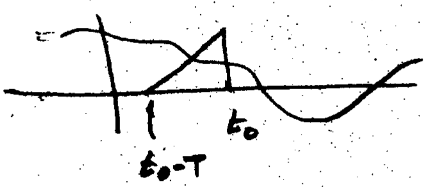


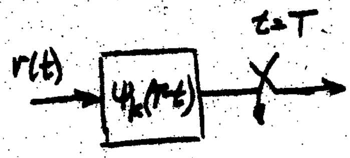
$y(0) = \int r(t)\, \psi_k^*(t)\, dt$ as desired.

Now make it causal with enough delay (how much is enough?)



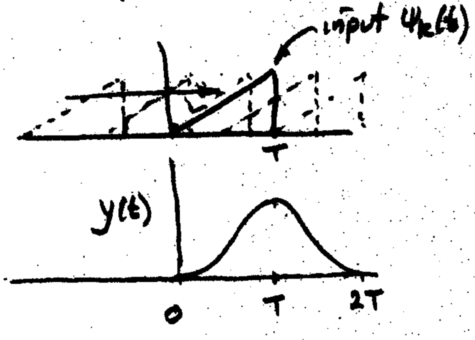
$y(t_0) = \int r(t)\, \psi_k^*(t - t_0 + T)\, dt$





$y(T) = \int r(t)\, \psi_k^*(t)\, dt$

The response to $\psi_k(t)$ itself is the pulse autocorrelation (delayed)



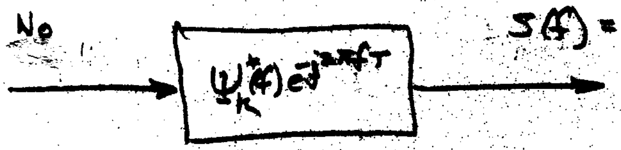
input $\Psi_k(f)$

filter $H(f) = \Psi_k^*(f)\, e^{-j2\pi f T}$

output is

$Y(f) = |\Psi_k(f)|^2 e^{-j2\pi f T}$

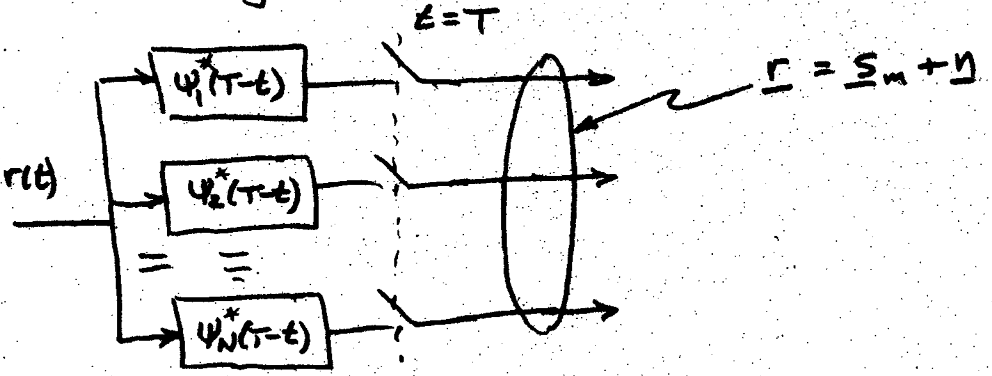
The autocorrelation function of the noise at the MF output is proportional to pulse autocorrelation function:



$S(f) = |H(f)|^2 N_0 = N_0 |\Psi_k(f)|^2$

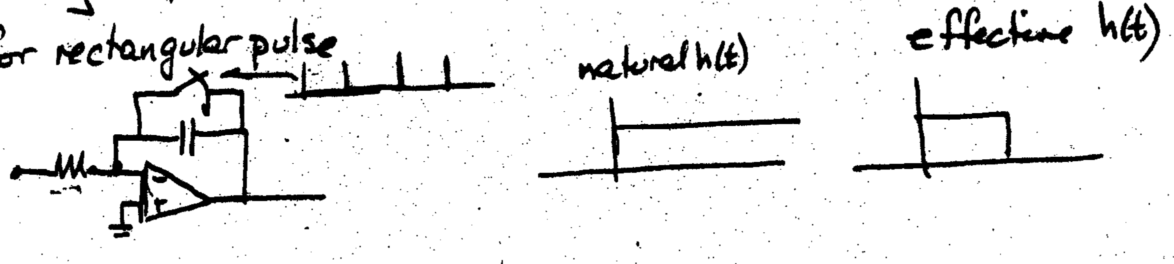
Hence autocorrelation function is $R(\tau) = N_0 R_{\psi_k}(\tau)$

- "Vectorising" with matched filters

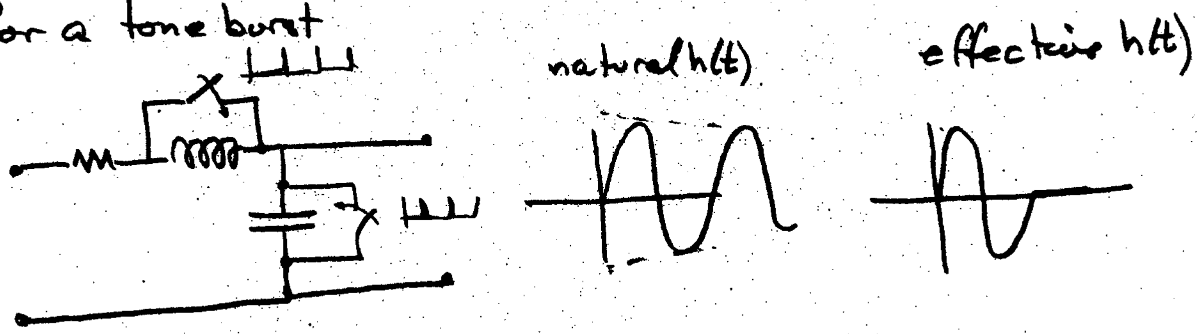


- Analog implementations – "integrate and dump"

for rectangular pulse



for a tone burst



END OF THE "ASIDE"

Making Decisions

see also your undergrad text or Proakis

- We now have a length-N vector $\underline{r} = \underline{s}_m + \underline{n}$ as a set of sufficient statistics — but what do we do with it?

- First, we determine the statistics. If we work with $\{\psi_k(t)\}$, the orthonormal basis, then the noise components satisfy $\frac{1}{2}\overline{n_k n_m^*} = N_0 \delta_{km}$.

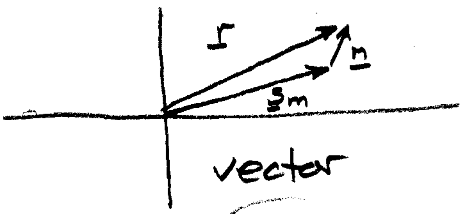
  That is, $R_n = \frac{1}{2}\overline{\underline{n}\,\underline{n}^\dagger} = N_0 I$

  Assuming Gaussian noise

$$p_{\underline{n}}(\underline{n}) = \frac{1}{(2\pi)^N |R_n|} \exp\left(-\frac{1}{2}\underline{n}^\dagger R_n^{-1}\underline{n}\right) = \frac{1}{(2\pi N_0)^N}\exp\left(-\frac{1}{2N_0}\underline{n}^\dagger\underline{n}\right)$$

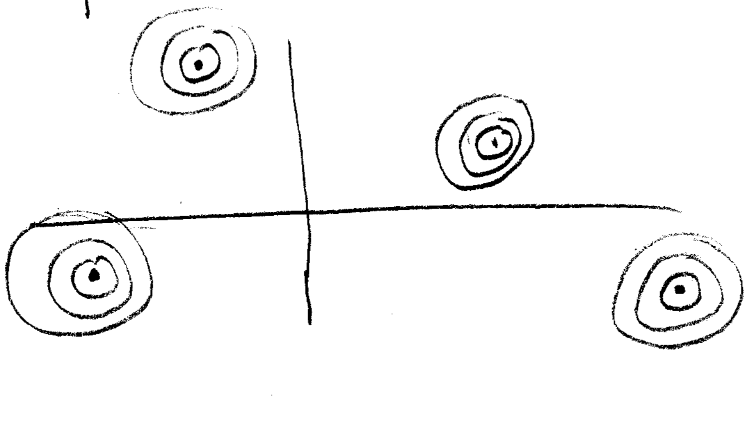
$$= \frac{1}{(2\pi N_0)^N}\exp\left(-\frac{1}{2N_0}\sum_{k=1}^{N}|n_k|^2\right) = \prod_{k=1}^{N}\left(\frac{1}{2\pi N_0} e^{-|n_k|^2/2N_0}\right)$$

Pictures (conceptual, since complex noise)



vector

iso-prob contours



with other signals

- Maximum a posteriori probability (MAP) decisions are easily formulated. Given some received $\underline{r}$, pick the signal that maximises

$$\max_i P(\underline{s}_i \mid \underline{r})$$

or $$\max_i \frac{p(\underline{s}_i, \underline{r})}{p(\underline{r})} = \frac{p(\underline{r} \mid \underline{s}_i) P(\underline{s}_i)}{p(\underline{r})}$$ now ignore $p(\underline{r})$ because it doesn't depend on i

or $$\max_i p(\underline{r} \mid \underline{s}_i) P(\underline{s}_i)$$ next, use $\underline{r} = \underline{s}_i + \underline{n}$

or $$\max_i p_{\underline{n}}(\underline{r} - \underline{s}_i) P(\underline{s}_i)$$

or $$\max_i \frac{1}{(2\pi N_0)^N} \exp\left(-\frac{1}{2N_0} \|\underline{r} - \underline{s}_i\|^2\right) P(\underline{s}_i)$$ now ignore const, take log

or $$\max_i -\|\underline{r} - \underline{s}_i\|^2 + 2N_0 \ln(P(\underline{s}_i))$$ expand

or $$\max_i -\left(\|\underline{r}\|^2 - 2\,\mathrm{Re}[\underline{s}_i^\dagger \underline{r}] + \|\underline{s}_i\|^2\right) + 2N_0 \ln(P(\underline{s}_i))$$ ignore $\|\underline{r}\|^2$ note $\|\underline{s}_i\|^2 = 2\mathcal{E}_i$

or $$\max_i \underbrace{\mathrm{Re}[\underline{s}_i^\dagger \underline{r}] - \underbrace{(\mathcal{E}_i - N_0 \ln(P(\underline{s}_i)))}_{a_i \text{ bias}}}_{\text{the "metric" (log likelihood) of } \underline{s}_i}$$

so the MAP detector calculates

$$\hat{i} = \arg\max_i \mathrm{Re}[\underline{s}_i^\dagger \underline{r}] - a_i$$

In the special case of equiprobable, equal energy signals, a_i is the same for all, so discard it. Just maximise the correlation $\mathrm{Re}[\underline{s}_i^\dagger \underline{r}]$.

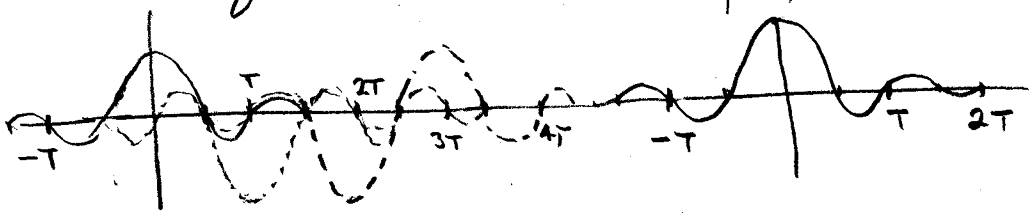
For further interpretation in case of real signals, see ENSC 428 notes, Section 5.3

Exploiting the Theorem of Irrelevance

- Theorem of Irrelevance again: in AWGN, the projection of the received waveform onto the signal space retains all the information necessary for a decision. We saw, for isolated pulse transmission, that it renders entire waveforms down to vectors.

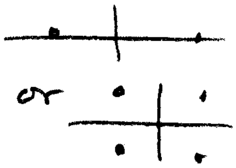
  Here we extend that thinking to unfamiliar situations; in all cases, we obtain sufficient statistics. So, regardless of what we do with vectors subsequently, we haven't lost any information by "vectorising".

  – Data sequence.



$$s(t) = \sum_k b(k)\, p(t-kT)$$

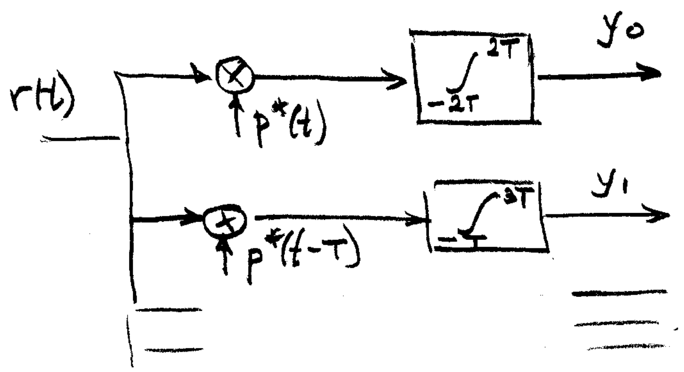
b(k) from constellation



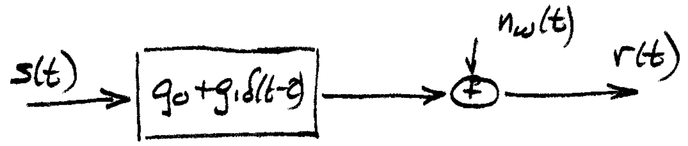
etc

Signal space is spanned by $p(t)$ and its translates $p(t-T)$, $p(t-2T)$ etc

Recover sufficient information for decisions from $r(t) = s(t) + n_w(t)$ by correlating against basis waveforms



– Received with echoes



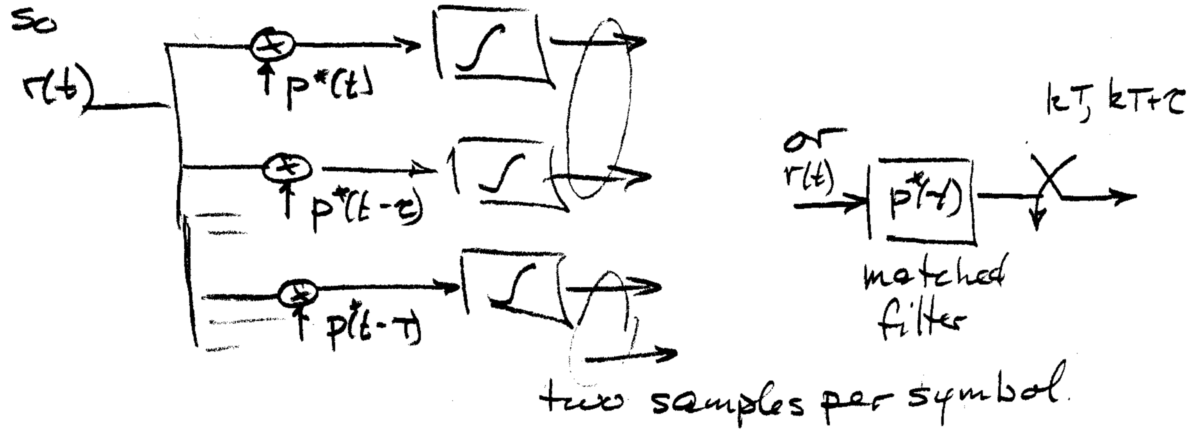
Received signal $g_0 s(t) + g_1 s(t-\tau)$

$$= g_0 \sum_k b(k)\, p(t-kT) + g_1 \sum_k b(k)\, p(t-\tau-kT)$$

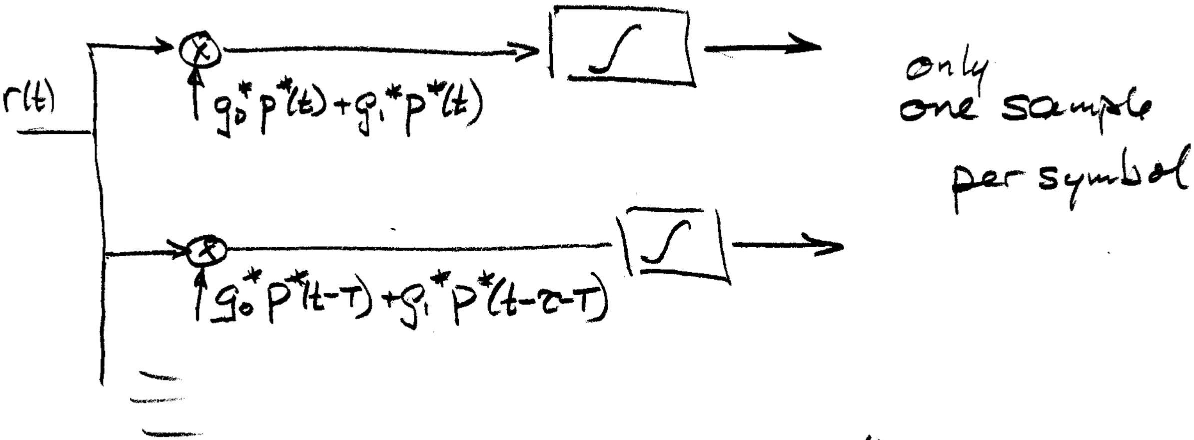
Signal space is spanned by

$$\{ p(t),\ p(t-\tau),\ p(t-T),\ p(t-\tau-T),\ \cdots\ p(t-iT),\ p(t-\tau-iT) \cdots \}$$

so



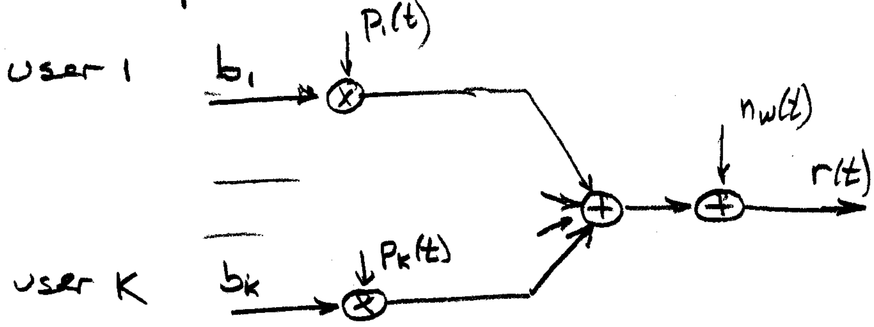
– Again, but note signal space is also spanned by $g_0 p(t) + g_1 p(t-\tau),\ g_0 p(t-T) + g_1 p(t-\tau-T), \cdots\cdots$



Often called a "Rake receiver".

This also falls out of ML analysis, but here we've got it just by observation.

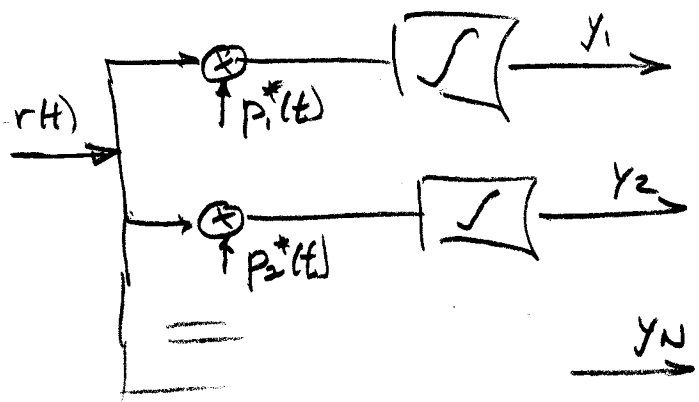
– Several users, each with a different pulse shape



$$r(t) = \sum_{k=1}^{K} b_k \, p_k(t) + n_w(t)$$

Signal space is spanned by $\{p_1(t), p_2(t) \cdots, p_K(t)\}$

So sufficient statistics obtained by

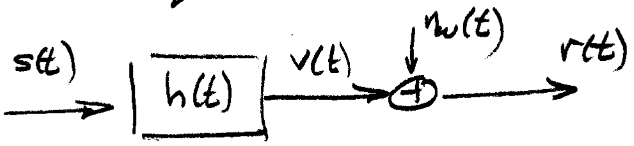


What if all users had same pulse shape?

What if users had different pulse shapes, but each went through its own channel with an echo?

2.3.11

– Data sequence through arbitrary channel



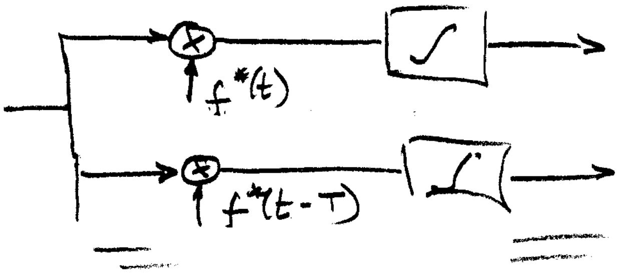
$s(t) = \sum_k b(k)\, p(t-kT)$

This is a continuous version of the echo channel.

$v(t) = s(t) \circledast h(t) = \sum_k b(k)\, f(t-kT)$

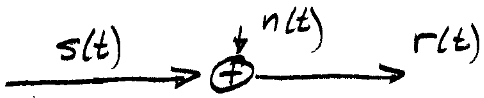
where $f(t) = p(t) \circledast h(t) = \int p(\alpha)\, h(t-\alpha)\, d\alpha$

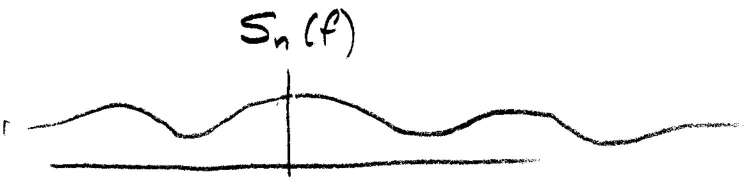
Received signal space is spanned by $\{f(t), f(t-T), \dots\}$



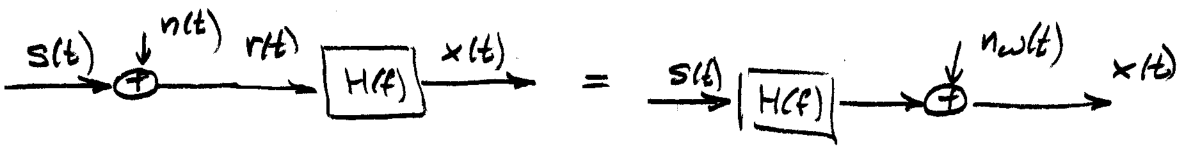
a more general form of the Rake receiver

– Coloured noise

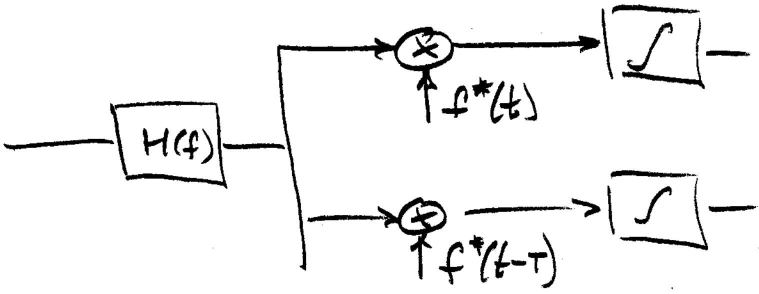




whiten the noise with an invertible filter (invertible so no information lost) $\quad H(f) = \frac{1}{\sqrt{S_n(f)}}$



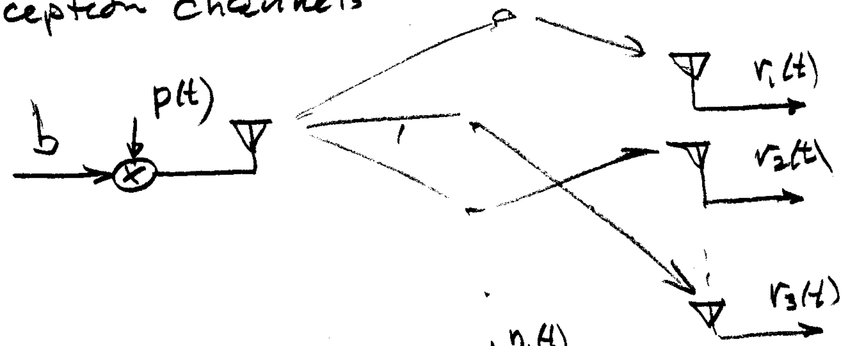
and process as above

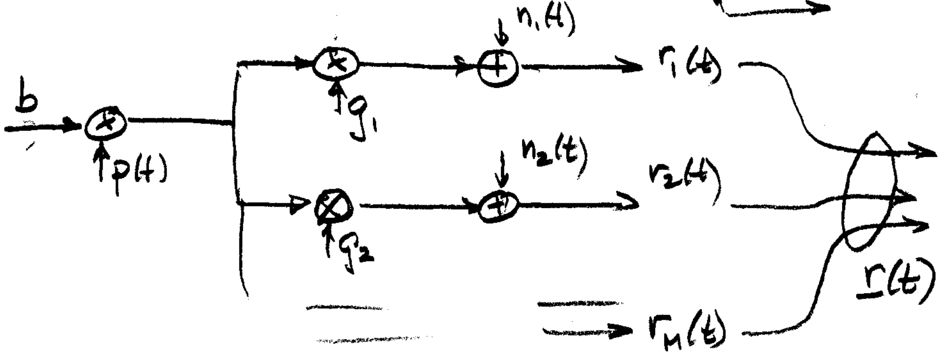


$f(t) = p(t) \circledast h(t)$

2.3.12

– Single user, isolated pulse, multiple reception channels

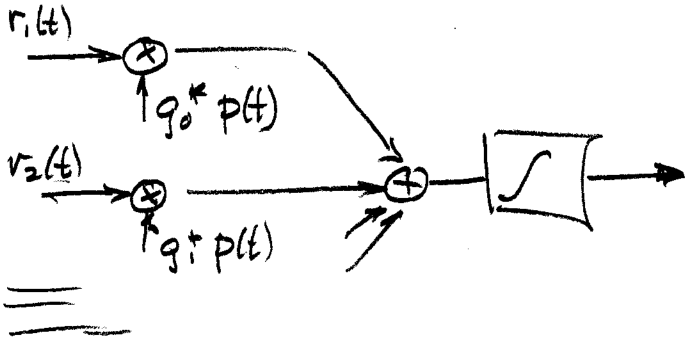




signal space is spanned by

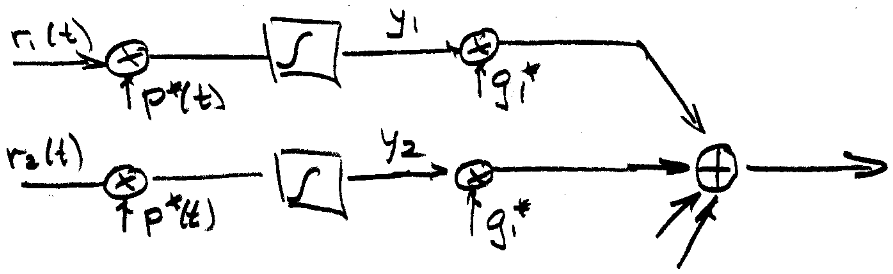
$$\begin{bmatrix} g_0\, p(t) \\ g_1\, p(t) \\ \vdots \\ g_M\, p(t) \end{bmatrix}$$

so correlate against it



a scalar!

or



This is "maximal ratio combining" receiver.