SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 428-4 Data Communications

Assignment 2 **Due: 2001 02 06**

1. Sum of Correlated Random Variables

Suppose we sum *N* equispaced samples of a zero-mean stationary process $x(t)$ with autocorrelation function $R_x(\tau)$ to form

$$y = \sum_{i=1}^{N} x_i$$

In general, the samples will be correlated, so the variance σ_y^2 does not necessarily equal the sum of the individual variances. However, the variance may still be asymptotically proportional to *N*, which can be useful. In this question, you will demonstrate that proportionality.

(a) Consider the samples to be stacked in a vector **x**. What is its covariance matrix **C**, and what pattern does it have? If $R_x(\tau)$ is negligible for $|\tau|$ greater than a few sample times, what does the matrix look like for large *N*?

(b) Represent the sum as $y = \mathbf{e}^T \mathbf{x}$ where **e** is the all-ones vector. Again assuming that the autocorrelation function is negligible after a few sample times, show that the variance σ_y^2 is asymptotically proportional to *N*, for large *N*.

2. Maximization of SNR

In Section 3 of the notes, we examined detection of the data bit $a = \pm 1$ from the sequence of *N* i.i.d. random variables

$$x_i = am + n_i$$

where *m*, the absolute value of the mean, is the same for all samples and the variance of all noise samples was the same, at σ^2. We concluded that the sensible thing was to add all the samples and examine the polarity of the sum. In this question, you will extend the result. The expressions you obtain are widely used in communications and signal processing.

(a) Suppose first that the pulses are not rectangular (but are still confined to a single bit duration). That is,

$$x_i = am_i + n_i$$

and the noise variances are still all the same. Clearly, our intuitive notion of adding or averaging the samples isn't the best approach – why should a very weak sample (small

m_i) be given the same importance as a strong one (large m_i)? So generalize to a weighted sum:

$$d = \sum_{i=1}^{N} w_i x_i$$

However, you still have to decide on what basis you should choose the weights. One way is to note that the signal component is the mean m_d and then maximize the ratio m_d / σ_d. Equivalently, maximize the square of the ratio (the SNR)

$$\gamma = \frac{m_d^2}{\sigma_d^2}$$

Write the SNR explicitly in terms of w_i, m_i and σ^2, then maximize it with respect to the weights. The Schwarz inequality is one way, but I want you to do it as a constrained maximization: maximize the numerator while constraining the denominator to have a particular value, using a Lagrange multiplier to do so.

(b) Now allow both the means m_i and variances σ_i^2 of the samples to depend on the sample index *i*. Find the maximizing set of weights the same way. Check your result: is it dimensionally consistent? What you should have obtained is an optimization that we can use in time varying systems, in interpreting a matched filter in the frequency domain with coloured noise and in developing "diversity combining" of antennas in radio systems.

3. Operation at Threshold (Think through this one, but don't hand it in.)

System designers often choose the parameters so that an FM or PCM system operates just above threshold (or if the receive power level varies, they allow an appropriate safety margin. Why is this a reasonable choice?.

4. A Repeater Chain

Here's the situation. You have to transmit a signal of bandwidth $B = 1\text{MHz}$ from Vancouver to Hope, a distance of about 150 km, over a cable for which the loss is 20 dB per km. Clearly, you need repeaters – or regenerators, since you haven't yet decided on FM or PCM. The amplifiers are set up to supply an output signal power of 10 watts (plus noise power), and the gain is adjusted to make up the attenuation in the preceding section of cable. The noise, modeled as an additive source at the input of the amplifier, has a PSD $N_o/2$= 100 picowatt/Hz. You are also fortunate to be able to use as much bandwidth as you want.

Now to decide on FM or PCM. An obvious parameter of interest is *M*, the required number of amplifiers, as well as the resulting SNR_o at the end of the chain. Assume that you adjust each candidate system (through modulation index β or number of bits n) so that it operates at threshold. Assume 3σ loading.

(a) For FM, how should β vary with *M*? Given this variation, how does the final SNR_o vary with *M*? Calculate it for a reasonable range of values for *M*. Is there an optimum value for *M*? Is even the optimized SNR_o an acceptable value? Define threshold conventionally as a 10 dB carrier power to noise power ratio.

(b) For PCM, how should *n* vary with *M*? Given this variation, how does the final SNR_o vary with *M*? Calculate it for a reasonable range of values for *M*. Is there an optimum value for *M*? Is the optimum a reasonable operating point? Define threshold as a 1 dB drop (i.e., 80%) from the saturated value of SNR_o. [Suggestions: Approximate the *Q* function with its simple overbound. Take the log of the threshold condition, and you get a simple quadratic equation in *n*. Approximate *n* as continuous and solve the quadratic.]

Notes:

1. Remember to calculate SNR at the INPUT of each amplifier, right after the noise is added. The following ideal amplification affects signal and noise equally, and won't change the SNR.

2. The problem is a little unrealistic, in that one would not invest in a system like this for a single 1 MHz signal. It would make more sense for a set of FDM carriers, but I wanted to keep it uncomplicated.