SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 428 Data Communications

Solutions to Assignment 1 **January 2001**

1. PDF of Max Value

The key to this problem is the observation that, for *y* to be less than or equal to some value, all of the inputs must be less than or equal to the same value:

$$y \le \eta \quad \text{implies} \quad x_i \le \eta, \quad \forall i$$

Since the inputs are statistically independent, the probability that all of them satisfy this condition is the product of the individual probabilities. Since they have the same distribution, we have

$$F_y(\eta) = F_x(\eta)^N$$

Differentiation gives the required pdf of *y*

$$f_y(\eta) = N\, F_x(\eta)^{N-1} f_x(\eta)$$

It is just as simple to work with the min function. For intermediate values, such as the 3rd largest, it's a little more complicated. If you need to know more about this type of problem (and it comes up from time to time), consult probability and statistics books for *rank order statistics*.

2. Conditional Gaussian PDF

For zero mean Gaussian random variables, the joint pdf has the form

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2\left(1-\rho^2\right)} \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - 2\rho \frac{x}{\sigma_x}\frac{y}{\sigma_y} \right) \right\}$$

We can obtain the conditional pdf $f_{Y|X}(y|x)$ as

$$f_{Y|X}(y|x) = \frac{f_{YX}(y,x)}{f_X(x)}$$

For this, we need the marginal pdf $f_X(x) = \int f_{XY}(x,y)dy$. To prepare for the integration over y, we rewrite the joint pdf by completing the square in the exponent:

$$f_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{y^2}{\sigma_y^2}-2\rho\frac{x}{\sigma_x}\frac{y}{\sigma_y}+\rho^2\frac{x^2}{\sigma_x^2}-\rho^2\frac{x^2}{\sigma_x^2}+\frac{x^2}{\sigma_x^2}\right)\right\}$$

$$=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_x^2}\right\}\exp\left\{-\frac{1}{2}\left(\frac{y-\rho\frac{\sigma_y}{\sigma_x}x}{\sigma_y\sqrt{1-\rho^2}}\right)^2\right\}$$

Using the normalizing factor on page 2.2.1 of the notes, we perform the integral over y to obtain the marginal pdf

$$f_X(x)=\frac{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\frac{x^2}{\sigma_x^2}\right\}=\frac{1}{\sqrt{2\pi}\sigma_x}\exp\left\{-\frac{x^2}{2\sigma_x^2}\right\}$$

So the marginal pdfs obtained from a joint Gaussian pdf are also Gaussian. Finally, we form the conditional pdf by division to obtain

$$f_{Y|X}(y|x)=\frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2}\left(\frac{y-\rho\frac{\sigma_y}{\sigma_x}x}{\sigma_y\sqrt{1-\rho^2}}\right)^2\right\}$$

(a) The conditional pdf is also Gaussian in y, as can be seen by inspection of the final result.

(b) The conditional mean is

$$m_{Y|X}=\rho\frac{\sigma_y}{\sigma_x}x=\frac{\sigma_{xy}^2}{\sigma_x^2}x$$

The expression makes sense: if the random variables are uncorrelated ($\rho=0$), then the best estimate is zero, the unconditional mean of y; if they are perfectly correlated ($\rho=1$), then the two variables are simply scaled by their respective variances; similarly with anticorrelation ($\rho=-1$). It also makes sense dimensionally when X and Y have different units. Note that x and the estimate of y are linearly related – it is a linear estimate.

This is an enormously important result. The conditional mean is the best estimate of Y, given X (best in the minimum mean squared error sense. For example, if you want to predict the value of a random process using the present value, and the process has autocorrelation function $R(\tau)$, then the best estimate is

$$\hat{x}(t+t_o) = \frac{R(t_o)}{R(0)} x(t)$$

(c) The conditional variance can also be read from the conditional pdf:

$$\sigma_{Y|X}^2 = \sigma_Y^2 \left(1-\rho^2\right)$$

It also makes sense: if the random variables are uncorrelated ($\rho = 0$), then the variance is just that of Y in its marginal pdf (no information from X); if they are perfectly correlated ($\rho = 1$) or perfectly anticorrelated ($\rho = -1$), then knowledge of X reveals everything about Y, and the conditional variance is zero.

3. Sum of Gaussian Random Variables

The two random variables have pdfs given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(-\frac{x^2}{2\sigma_X^2}\right) \quad \text{and} \quad \mathrm{f}_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{y^2}{2\sigma_Y^2}\right)$$

(a) The characteristic function of a sum of independent variables $z = x + y$ is the product of the individual characteristic functions, so

$$M_Z(\omega) = M_X(\omega) M_Y(\omega) = \exp\left(-\tfrac{1}{2}\omega^2\sigma_X^2\right)\exp\left(-\tfrac{1}{2}\omega^2\sigma_Y^2\right) = \exp\left(-\tfrac{1}{2}\omega^2(\sigma_X^2 + \sigma_Y^2)\right)$$

The product has the form of a characteristic function of a Gaussian random variable with variance equal to the sum of the variances.

(b) To obtain the pdf of the sum by a direct convolution, we express it as

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-\alpha) f_Y(\alpha) d\alpha$$

$$= \frac{1}{2\pi\sigma_X\sigma_Y} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{z-\alpha}{\sigma_X}\right)^2\right\} \exp\left\{-\frac{1}{2}\frac{\alpha^2}{\sigma_Y^2}\right\} d\alpha$$

I won't carry it through, but you can see how it will go. As in Question 2, complete the square and use the normalization to ease the pain of integration.