SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 428 Data Communications

Solutions to Assignment 3 **February 2001**

1. Unequal Prior Probabilities

Binary antipodal signals form a one-dimensional constellation, so r, s_1 and s_2 are scalars.

(a) The signals s_1 = +sqrt(E_b) and s_2 = -sqrt(E_b) form the means of the two conditional pdfs. They both have variance $N_o/2$, so the conditional pdfs are

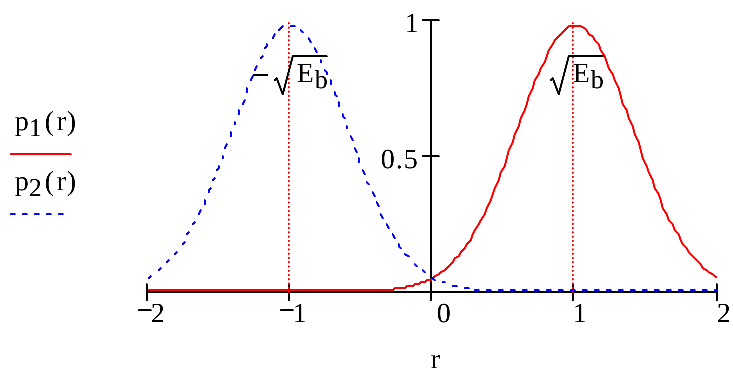
$$p_{r|s1}(r|s_1) = \frac{1}{\sqrt{\pi N_o}} \cdot \exp\left[-\frac{\left(r-\sqrt{E_b}\right)^2}{N_o}\right] \qquad p_{r|s1}(r|s_2) = \frac{1}{\sqrt{\pi N_o}} \cdot \exp\left[-\frac{\left(r+\sqrt{E_b}\right)^2}{N_o}\right]$$

To sketch them, assign arbitrary values

$$E_b := 1 \qquad \gamma_b := 3 \qquad N_o := \frac{E_b}{\gamma_b} \qquad s_1 := \sqrt{E_b} \qquad s_2 := -\sqrt{E_b}$$ and define

$$p_1(r) := \frac{1}{\sqrt{\pi N_o}} \cdot \exp\left[-\frac{\left(r-\sqrt{E_b}\right)^2}{N_o}\right] \qquad p_2(r) := \frac{1}{\sqrt{\pi N_o}} \cdot \exp\left[-\frac{\left(r+\sqrt{E_b}\right)^2}{N_o}\right]$$

$$r := -2\cdot\sqrt{E_b}, -1.98\cdot\sqrt{E_b} .. 2\cdot\sqrt{E_b}$$

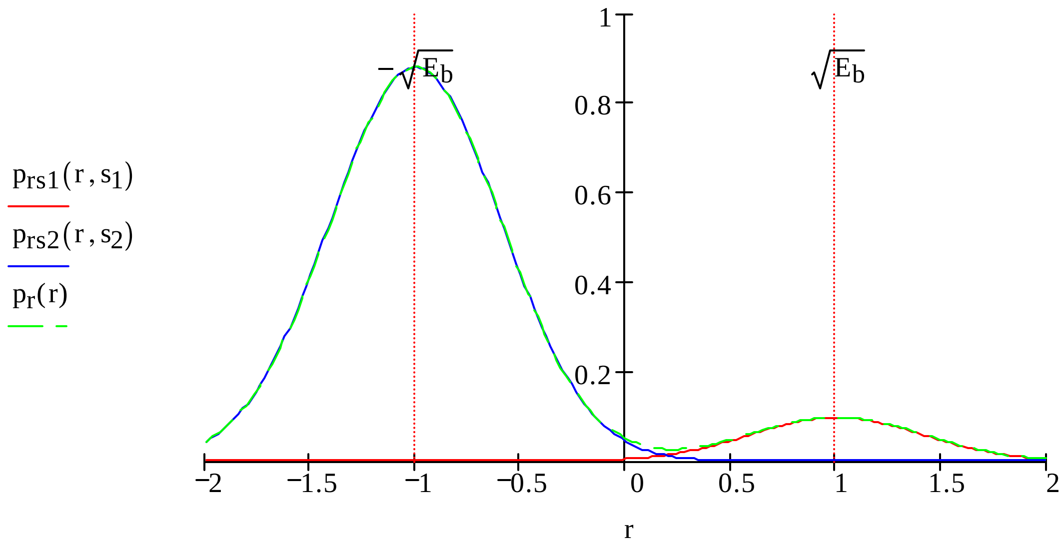


The *conditional* pdfs do not depend on the prior probabilities.

Now assign some arbitrary prior probabilities $P_1 := 0.1$ $P_2 := 1 - P_1$

The joint probabilities and the marginal probability are

$$p_{rs1}(r, s_1) := P_1 \cdot p_1(r) \qquad p_{rs2}(r, s_2) := P_2 \cdot p_2(r) \qquad p_r(r) := p_{rs1}(r, s_1) + p_{rs2}(r, s_2)$$



(b) The decision boundary is located where the joint probabilities are equal. From page 5.3.7 of the notes, this requires the threshold *r* to satisfy

$$s_1 \cdot r + \frac{N_o}{2} \cdot \ln(P_1) = s_2 \cdot r + \frac{N_o}{2} \cdot \ln(P_2)$$

since the signals have equal power

or, scaling the threshold by the signal amplitude,

$$\frac{r_{thresh}}{\sqrt{E_b}} = \frac{1}{4 \cdot \gamma_b} \cdot \ln\left(\frac{P_2}{P_1}\right)$$

The boundary is a function of the prior probability ratio and, if the probabilities are unequal, it also depends on the SNR. Large disparity between the priors can even shift the boundary beyond one of the signal amplitudes.

(c) The *conditional* error probabilities are depend on the distance to the threshold, measured in standard deviations

$$\Pr(\text{error} \mid s_1) = Q\left(\frac{\sqrt{E_b} - r_{thresh}}{\sqrt{\frac{N_o}{2}}}\right) = Q\left[\sqrt{2\cdot\gamma_b}\cdot\left(1 - \frac{1}{4\cdot\gamma_b}\cdot\ln\left(\frac{P_2}{P_1}\right)\right)\right]$$

$$\Pr(\text{error} \mid s_2) = Q\left(\frac{r_{thresh} + \sqrt{E_b}}{\sqrt{\frac{N_o}{2}}}\right) = Q\left[\sqrt{2\cdot\gamma_b}\cdot\left(1 + \frac{1}{4\cdot\gamma_b}\cdot\ln\left(\frac{P_2}{P_1}\right)\right)\right]$$

Note that the *Q* function is greater than 1/2 for negative arguments, and approaches 1 as the argument approaches -∞. Consequently, one of the conditional error probabilities can approach 1 for extreme disparity between the prior probability. The average error probability is

$$P_b = \Pr(\text{error} \mid s_1)\cdot P_1 + \Pr(\text{error} \mid s_2)\cdot P_2$$

$$= Q\left[\sqrt{2\cdot\gamma_b}\cdot\left(1 - \frac{1}{4\cdot\gamma_b}\cdot\ln\left(\frac{P_2}{P_1}\right)\right)\right]\cdot P_1 + Q\left[\sqrt{2\cdot\gamma_b}\cdot\left(1 + \frac{1}{4\cdot\gamma_b}\cdot\ln\left(\frac{P_2}{P_1}\right)\right)\right]\cdot P_2$$

2. Translation of Signal Constellations

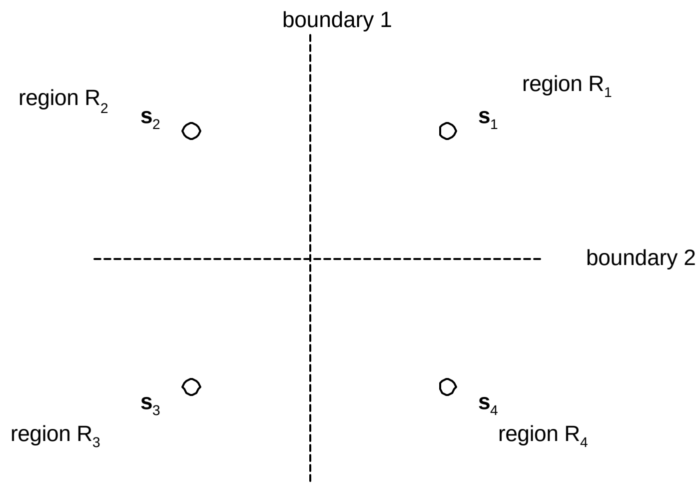
We noted in class the similarity of average energy with moment of inertia. Both are minimized by locating the centroid (the centre of mass) at the origin (the centre of rotation). The simplest proof is to consider a constellation that is already centred on the origin with average energy E_s. From page 5.4.2 of the notes, translation by a vector **l** produces new energy

$$E'_s = E_s + (|\mathbf{l}|)^2 + 2\cdot\mathbf{l}^T\cdot\mathbf{s_c} = E_s + (|\mathbf{l}|)^2$$

Thus the new energy is greater than the old, increasing with increasing distance, irrespective of direction, and centreing the constellation minimizes the energy.

3. Union Bound

The sketch below shows that the decision regions are the quadrants. Each signal is a distance $\sqrt{E_s}$ from the origin, and has components along each axis of $\sqrt{E_b}$ since $E_s = 2\cdot E_b$



(a) Define E_1 as the event that noise carries the received vector to the wrong side (the left side) of boundary 1, similarly for E_2. The probability of a symbol error is upper bounded by

$$\Pr(E_1 \cup E_2) \le \Pr(E_1) + \Pr(E_2) = 2 \cdot Q\left(\sqrt{2 \cdot \gamma_b}\right)$$

since the distance to the decision boundaries, measured in standard devations, is $\sqrt{\frac{2}{N_o} \cdot E_b}$

(b) The probability of E_1 is the probability that the received **r** lies in the left half plane and for E_1 it's the lower half plane. Consequently, Region 3 is counted twice. That's why it is an upper bound on the probability of the union. It is exact only if the events are mutually exclusive, so their regions do not overlap.

This constellation is easy to work with, since its decision regions are all rectangular (although semi-infinite). Since the noise components are independent,

$$\Pr(\text{Region3}) = Q\left(\sqrt{2 \cdot \gamma_b}\right)^2$$

For most problems in which we resort to the union bound, it is virtually impossible to calculate the probabilities of overlap regions. It's often hard even to describe what those regions are.

(c) To fix up the bound, we subtract the double counted region

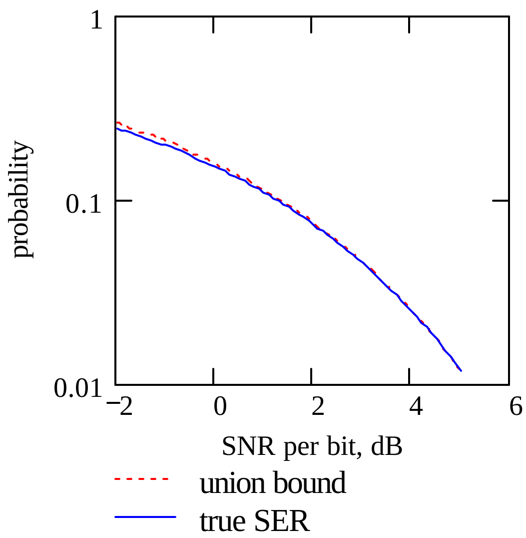
$$\Pr(E_1 \cup E_2) = \Pr(E_1) + \Pr(E_2) - \Pr(E_1 \cap E_2) = 2 \cdot Q\left(\sqrt{2 \cdot \gamma_b}\right) - Q\left(\sqrt{2 \cdot \gamma_b}\right)^2$$

or

$$P_s(\gamma_b) := 2\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)\cdot\left(1-\frac{1}{2}\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)\right) \qquad P_{bound}(\gamma_b) := 2\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)$$

The second term in parentheses represents the relative effect of the correction, and it goes to zero with increasing SNR, meaning that the union bound becomes tight at high SNR. Let's see it graphically:

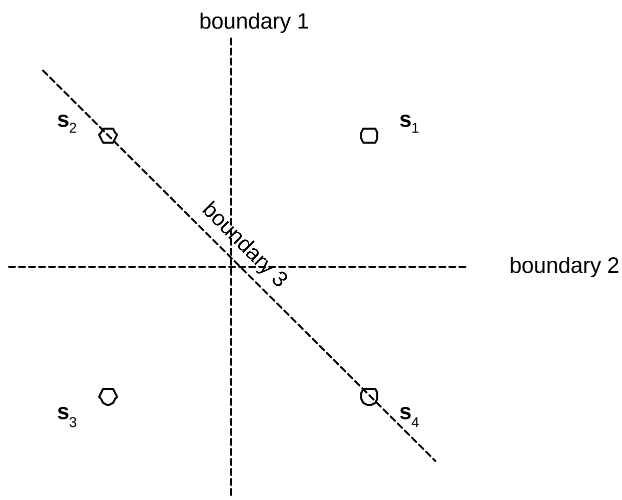
$$\gamma_{bdB} := -2, -1.9 .. 5$$



The bound is tight enough for any error rate that's good enough to use.

I won't bother sketching the probability surface.

(d) If the events in question are the three pairwise error events, the boundaries are as shown



This form of the union bound can be written

$$P_s \le P_2(s_2) + P_2(s_3) + P_2(s_4) = Q\left(\sqrt{2\cdot\gamma_b}\right) + Q\left(\sqrt{4\cdot\gamma_b}\right) + Q\left(\sqrt{2\cdot\gamma_b}\right)$$

or

$$P_{bound} = 2\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)\cdot\left(1 + \frac{1}{2}\cdot\frac{Q\left(\sqrt{4\cdot\gamma_b}\right)}{Q\left(\sqrt{2\cdot\gamma_b}\right)}\right)$$

It is looser than the earlier bound $2\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)$ because it uses a decision boundary in addition to the nearest ones, even though only the nearest ones really define the error. That is, if it's over boundary 3, then it's over at least one of the other boundaries.

The bound's second term in the parentheses decreases to zero quickly. In fact the ratio can be approximated by use of the approximation to the Q function:

$$\frac{Q\left(\sqrt{4\cdot\gamma_b}\right)}{Q\left(\sqrt{2\cdot\gamma_b}\right)} = \frac{\exp\left(\frac{-1}{2}\cdot 4\cdot\gamma_b\right)}{\exp\left(\frac{-1}{2}\cdot 2\cdot\gamma_b\right)} = \exp(-\gamma_b) \quad \text{approximately}$$

so the two forms of the union bound converge at higher SNR. The superfluous term in the second form of the bound represents a point that is farther away by a factor sqrt(2), so its effect becomes negligible compared with that of the closer points as SNR increases.

4. Gram-Schmidt

These three functions span the space:

$$v_0(t) = 1 \qquad v_1(t) = t \qquad v_2(t) = t^2 \qquad \text{for} \quad \frac{-1}{2} \le t \le \frac{1}{2}$$

We can do G-S on them in any order, but I'll do it in the obvious way.

Step 0: Just normalize v_0. $\quad u_0(t) := 1$ has unit energy over $\frac{-1}{2} \le t \le \frac{1}{2}$

Step 1: Project v1 onto u0

$$\int_{-0.5}^{0.5} t\cdot u_0(t)\,dt = 0 \qquad v_{1hat}(t) := 0$$

The error is $e_1(t) = v_1(t) - v_{1hat}(t) = t$ with energy $E_{e1} = \int_{-0.5}^{0.5} t^2\,dt = \frac{1}{12}$

so the next orthonormal basis function is obtained by normalizing e1:

$u_1(t) := \sqrt{12}\cdot t$ which has unit energy over $\frac{-1}{2} \le t \le \frac{1}{2}$

Step 2: Project v2 onto u0 and u1:

$$\int_{-0.5}^{0.5} t^2 \cdot u_0(t)\,dt = \frac{1}{12} \qquad \int_{-0.5}^{0.5} t^2 \cdot u_1(t)\,dt = 0$$

and the approximation is $v_{2hat}(t) = \frac{1}{12}\cdot u_0(t) + 0\cdot u_1(t) = \frac{1}{12}$

which has error $e_2(t) = v_2(t) - v_{2hat}(t) = t^2 - \frac{1}{12}$

with energy $$E_{e2} = \int_{-0.5}^{0.5} \left(t^2 - \frac{1}{12}\right)^2 dt = \frac{1}{180}$$

so our third orthonormal function is $u_2(t) := \frac{30}{\sqrt{5}}\cdot\left(t^2 - \frac{1}{12}\right)$

Plot them $t := -0.5, -0.49 .. 0.5$