SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 428 Data Communications

Solutions to Assignment 4 **March 2001**

1. How Important is a Matched Filter?

(a) First, we'll do the matched filter receiver. For this, we can simply set

$$g_R(t) = g_T(t) = A\cdot\sin\left(\pi\cdot\frac{t}{T}\right) \quad 0 \le t \le T$$

The signal amplitude in the matched filter output sample is

$$\mu = \int_0^T g_R(t)\cdot g_T(t)\,dt = A^2\cdot\int_0^T \sin\left(\pi\cdot\frac{t}{T}\right)^2 dt = \frac{A^2}{2}\cdot T$$

which also equals the energy per bit E_b . The noise variance in the sample is

$$\sigma^2 = \frac{N_o}{2}\cdot\int_0^T g_R(t)^2\,dt = \frac{N_o\cdot A^2\cdot T}{4}$$

so the BER for MF detection is the familiar $P_{bMF} = Q\left(\frac{\mu}{\sigma}\right) = Q\left(\sqrt{\frac{A^2\cdot T}{N_o}}\right) = Q\left(\sqrt{2\cdot\gamma_b}\right)$

Next, we do the integrator receiver. For this, we use

$$g_R(t) = 1 \quad 0 \le t \le T$$

The amplitude of the signal component of the integrator output is

$$\mu = \int_0^T g_T(t)\,dt = \frac{2}{\pi}\cdot A\cdot T$$

and the variance of the noise component is

$$\sigma^2 = \frac{N_o}{2} \cdot \int_0^T 1^2\, dt = \frac{N_o \cdot T}{2}$$

The BER is therefore $$P_{bint} = Q\left(\frac{\mu}{\sigma}\right) = Q\left(\sqrt{\frac{8 \cdot A^2 \cdot T}{\pi^2 \cdot N_o}}\right) = Q\left(\sqrt{\frac{16}{\pi^2} \cdot \gamma_b}\right) = Q\left(\frac{4}{\pi} \cdot \sqrt{\gamma_b}\right)$$

The effective SNR is less with the integrator, and the cost is a factor of

$$2 \cdot \left(\frac{\pi^2}{16}\right) = \frac{\pi^2}{8}$$ or, in dB, $$\mathrm{dB}\left(\frac{\pi^2}{8}\right) = 0.912$$

(b) Now for the related question in the frequency domain. The transmitted pulse, in frequency, is

$$G_T(f) = A \cdot T \cdot \mathrm{rect}(f \cdot T)$$ where $$\mathrm{rect}(x) = \begin{cases} 1 & \text{if } \frac{-1}{2} \le x \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

or you could write it like this:

$$G_T(f) \begin{cases} A \cdot T & \text{if } |f| \le \frac{1}{2 \cdot T} \\ 0 & \text{otherwise} \end{cases}$$

A matched filter receiver might use a unit dc gain version of the transmitted pulse to make it look like the sketch in the question paper:

$$G_R(f) = \mathrm{rect}(f \cdot T)$$

The amplitude component in the filter output sample is (using Parseval's identity)

$$\mu = \int_0^T g_T(t) \cdot g_R(t)\, dt = \int_{-\infty}^{\infty} G_T(f) \cdot G_R(f)\, df = A$$

and the noise variance of the filter output is

$$\sigma^2 = \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} \left(\left| G_R(f) \right|\right)^2 df = \frac{N_o}{2 \cdot T}$$

From this, we have the BER $P_{bMF} = Q\left(\frac{\mu}{\sigma}\right) = Q\left(\sqrt{\frac{2 \cdot A^2 \cdot T}{N_o}}\right) = Q\left(\sqrt{2 \cdot \gamma_b}\right)$

since $E_b = \int_{-\infty}^{\infty} \left(\left| G_T(f) \right|\right)^2 df = A^2 \cdot T$ by Parseval.

Now we turn to the receiver filter with sidelobes. The signal component is unchanged, at $\mu = A$. The noise variance is

$$\sigma^2 = \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} \left(\left| G_R(f) \right|\right)^2 df = \frac{N_o}{2} \cdot \left(\frac{1}{T} + 2 \cdot \frac{d^2}{T}\right) = \frac{N_o}{2 \cdot T} \cdot \left(1 + 2 \cdot d^2\right)$$

which makes the BER

$$P_{bsidel} = Q\left(\frac{\mu}{\sigma}\right) = Q\left[\sqrt{\frac{2 \cdot A^2 \cdot T}{N_o \cdot \left(1 + d^2\right)}}\right] = Q\left(\sqrt{\frac{2}{1 + 2 \cdot d^2} \cdot \gamma_b}\right)$$

The cost in effective SNR is a factor

$$1 + 2 \cdot d^2 \qquad \text{or, in dB} \qquad 10 \cdot \log\left(1 + 2 \cdot d^2\right) = \frac{10}{\ln(10)} \cdot \ln\left(1 + 2 \cdot d^2\right) = \frac{20 \cdot d^2}{\ln(10)}$$

where the last "equality" is an approximation based on small sidelobes. Even if those sidelobes are only 10 dB down (d^2=0.1), the SNR penalty is only

$$\frac{20}{\ln(10)} \cdot 0.1 = 0.869 \quad \text{dB}$$

Depending on application, this may be a price well worth paying in order to use a simpler filter. For satellite applications, however, and particularly on the downlink, we would not willingly give up 1 dB.

2. Viterbi Receiver for FSK

(a) The signal space is spanned by $g_1(t)$, $g_2(t)$ and their translates by multiples of T. Since the two pulses are already orthogonal, we have an easy orthonormal basis set:

$$\psi_1(t) = \frac{\sqrt{2}}{A} \cdot g_1(t) \qquad \psi_2(t) = \frac{\sqrt{2}}{A} \cdot g_2(t)$$

Projection of $r(t)$ in $kT \le t \le (k+1)T$ onto the basis waveforms gives the vector

$$\mathbf{r}(k) = \begin{bmatrix} r_1(k) \\ r_2(k) \end{bmatrix} \qquad \text{where} \qquad r_i(k) = \int_0^T r(t + k \cdot T) \cdot \psi_i(t)\, dt$$

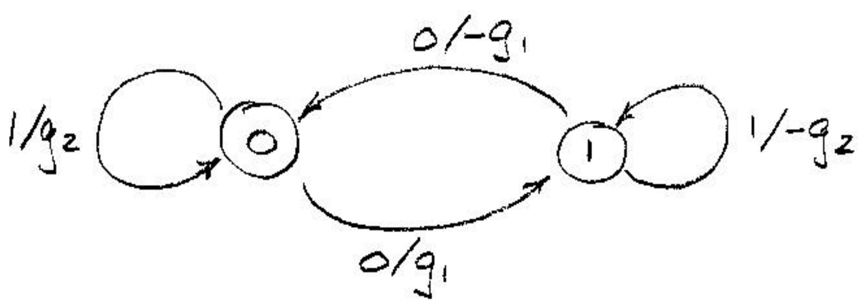
It is the sum of signal vector plus noise vector $\mathbf{r}(k) = \mathbf{s}(k) + \mathbf{n}(k)$

where the noise covariance matrix is $\mathbf{C}_\mathbf{n} = \frac{N_o}{2} \cdot \mathbf{I}_\mathbf{2}$ with $\mathbf{I}_2$ as the 2x2 identity matrix.

and the signal vectors are $\mathbf{s}_\mathbf{1} = \sqrt{E_b} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad \mathbf{s}_\mathbf{2} = \sqrt{E_b} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad$ with $\qquad E_b = \frac{A^2}{2} \cdot T$

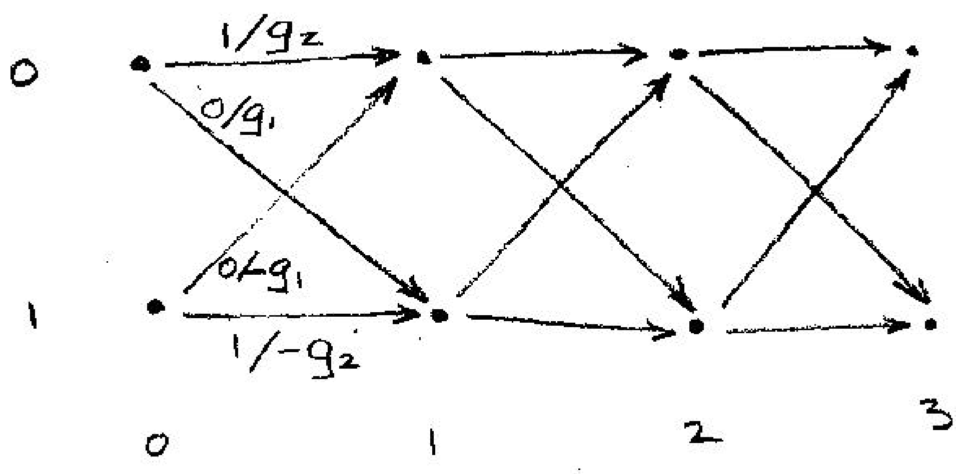
and their negatives $-\mathbf{s}_\mathbf{1}$ and $-\mathbf{s}_\mathbf{2}$

(b) The branches in the state diagram below are labeled "logic level/transmitted waveform". You can see that logic 0 causes g_1 to be transmitted, thereby reversing the starting slope, and logic 1 causes g_1 to be transmitted, so no slope change. The trellis unrolls the state diagram.



state 0:
start with positive slope

state 1:
start with negative slope



(c) It's not necessary for your answer, but, just for completeness, I'll back up a little in the derivation. Suppose we receive *N* symbols, and therefore $\mathbf{r}(k)$, k=1,,N . Stack those length-2 vectors into a complete received vector of length 2*N*

$$\mathbf{r}_{2N}=\begin{bmatrix} \mathbf{r}(N) \\ \mathbf{r}(N-1) \\ \vdots \\ \vdots \\ \mathbf{r}(2) \\ \mathbf{r}(1) \end{bmatrix} \qquad \text{so} \qquad \mathbf{r}_{2N}=\mathbf{s}_{2N}+\mathbf{n}_{2N}$$

$$\text{with noise covariance} \qquad \mathbf{C}_{n2N}=\frac{N_o}{2}\cdot\mathbf{I}_{2N}$$

The ML criterion is written below with commas only because it's hard to write a conditional probability in Mathcad. The expression for $\mathbf{C}_{n2N}$ has already been substituted in.

$$\operatorname{argmax}_{s2N}\left(p\left(\mathbf{r}_{2N},|,\mathbf{s}_{2N}\right)\right)=\operatorname{argmax}_{s2N}\left[\frac{1}{\left(\pi\cdot N_o\right)^N}\cdot\exp\left[\frac{-1}{N_o}\cdot\left(\left|\mathbf{r}_{2N}-\mathbf{s}_{2N}\right|\right)^2\right]\right]$$

$$=\operatorname{argmin}_{s2N}\left[\left(\left|\mathbf{r}_{2N}-\mathbf{s}_{2N}\right|\right)^2\right]=\operatorname{argmin}_{s2N}\left[\sum_{k=1}^{N}\left(\left|\mathbf{r}(k)-\mathbf{s}(k)\right|\right)^2\right]$$

We have now reduced it to a sum of *N* branch metrics.

Next, we recognize that there is a one-to-one correspondence between signal sequences $\mathbf{s}_{2N}$ and state sequences s_N, and write the branch metrics as

$$\mu(\mathbf{r},\sigma_{now},\sigma_{then}) = \left(\left\| \mathbf{r} - \mathbf{s}(\sigma_{now},\sigma_{then}) \right\|\right)^2$$

This expression, written out for each of the four transitions, is a perfectly good answer:

$$\mu(\mathbf{r},0,0) = \left(\left\| \mathbf{r} - \mathbf{s}_{\mathbf{2}} \right\|\right)^2 \qquad \mu(\mathbf{r},1,0) = \left(\left\| \mathbf{r} - \mathbf{s}_{\mathbf{1}} \right\|\right)^2$$

$$\mu(\mathbf{r},0,1) = \left(\left\| \mathbf{r} + \mathbf{s}_{\mathbf{1}} \right\|\right)^2 \qquad \mu(\mathbf{r},1,1) = \left(\left\| \mathbf{r} + \mathbf{s}_{\mathbf{2}} \right\|\right)^2$$

However, each one takes two subtractions (since two components), two squarings and an additions. We can simplify it to reduce computation, and gain some insight in the process. First, expand the quadratic in the branch metric

$$\mu(\mathbf{r},\sigma_{now},\sigma_{then}) = (\|\mathbf{r}\|)^2 - 2\cdot\mathbf{r}^T\cdot\mathbf{s}(\sigma_{now},\sigma_{then}) + \left(\left\| \mathbf{s}(\sigma_{now},\sigma_{then}) \right\|\right)^2$$

Drop the first term because it is common to all signals. The third term is E_b, again common to all signals, so drop it, too. Now we have a new expression for branch metric, one which we want to *maximize*, given by

$$\mu(\mathbf{r},\sigma_{now},\sigma_{then}) = \mathbf{r}^T\cdot\mathbf{s}(\sigma_{now},\sigma_{then})$$

Now substitute the signals from part (a) $\mathbf{s}_{\mathbf{1}} = \sqrt{E_b}\cdot\begin{bmatrix}1\\0\end{bmatrix}$ $\quad \mathbf{s}_{\mathbf{2}} = \sqrt{E_b}\cdot\begin{bmatrix}0\\1\end{bmatrix}$ to obtain

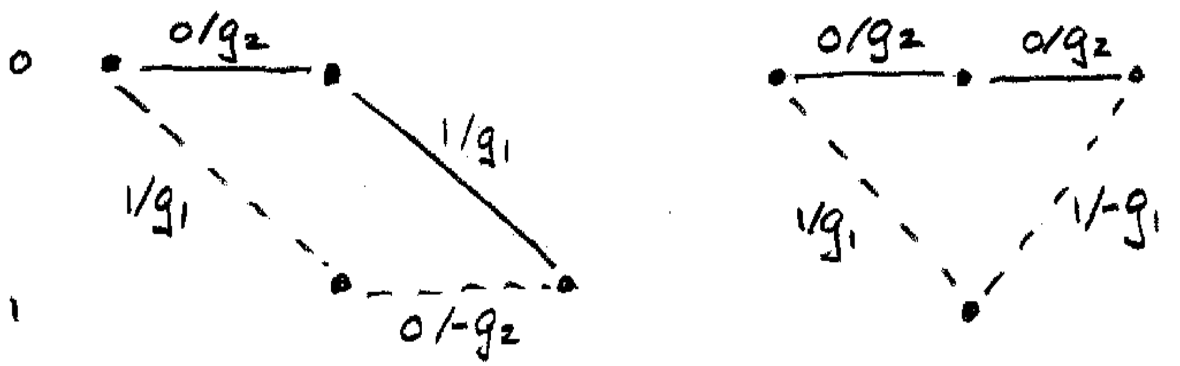
$$\mu(\mathbf{r},0,0) = r_2 \qquad \mu(\mathbf{r},1,0) = r_1$$

$$\mu(\mathbf{r},0,1) = -r_1 \qquad \mu(\mathbf{r},1,1) = -r_2$$

These require no computation at all, except for negation.

The metric for any overall sequence $\mathbf{s}_{2N}$ (or s_N) is just the correlation of the sequence with the receieved vector $\mathbf{r}_{2N}$, and this correlation equals the sum of the branch correlations. Because the pulse shapes are orthogonal, each branch correlation in turn is just the the appropriate filter output, negated if necessary.

(d) Inspection of the trellis shows that, unlike the AMI trellis, there are no branches that carry the same signal in a given symbol interval. Thus the Euclidean distance between alternative paths increases with length of an error event, which decreases the probability of that event. Consequently the shortest error events are most likely. They are of length 3 and are shown below. By symmetry, there is no loss of generality in starting them at state 0 or choosing a particular path in each pair to be the "correct" one.



There are two data bit errors in each of these events. Now that I look at it more closely, I realize that I could have reduced it to a single bit error by reversed the labeling on the branches emanating from state 1; that is, state 1 to state 0 carries logic 0, and state 1 to state 1 carries logic 1. Instead of associating a logical value with a frequency, I could have associated the value with whether the next pulse starts with positive slope. Darn - cutting the error rate in half for free is always worthwhile. On the other hand, my original labeling is at least resistant to an accidental reversal of the polarity of the signals (e.g., due to phase ambiguity in carrier recovery for bandpass signals, or simply getting wires reversed in installation).

Anyway, take the left hand event above (they both have the same pairwise probability, so it doesn't matter) and note that the nominal correct state sequence is 0,0,1. The received signals are then

$$\mathbf{r}(1) = \sqrt{E_b}\cdot\begin{bmatrix}0\\1\end{bmatrix} + \mathbf{n}(1) \qquad \mathbf{r}(2) = \sqrt{E_b}\cdot\begin{bmatrix}1\\0\end{bmatrix} + \mathbf{n}(2)$$

The pairwise probability of error is the probability that the metric of the true sequence minus the metric of the false sequence is negative, as developed in the following sequence of expressions:

$$\Pr(\mu(\mathbf{r}(1),0,0) + \mu(\mathbf{r}(2),1,0) - (\mu(\mathbf{r}(1),1,0) + \mu(\mathbf{r}(2),1,1))<0)$$

$$\Pr\left[r_2(1) + r_1(2) - \left(r_1(1) - r_2(2)\right)<0\right]$$

$$\Pr\left[\sqrt{E_b} + n_2(1) + \sqrt{E_b} + n_1(2) - \left(n_1(1) - n_2(2)\right) < 0\right]$$

$$\Pr\left(2\cdot\sqrt{E_b} + \nu < 0\right)$$ where $$\sigma_\nu^2 = 4\cdot\frac{N_o}{2}$$

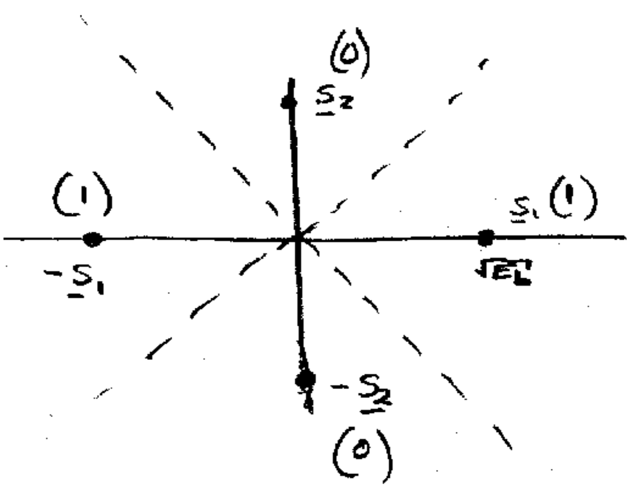
so, finally, the pairwise error probability is

$$P_2 = Q\left(\frac{2\cdot\sqrt{E_b}}{\sigma_\nu}\right) = Q\left(\sqrt{2\cdot\gamma_b}\right)$$

The question didn't ask for it, but you can see that it is easy to calculate the asymptotic error rate (i.e., if only the shortest events are significant). Each correct sequence has only a single incorrect counterpart, and they all have the same pairwise probabilities, so averaging over the correct sequences and counting two bit errors per event gives the BER

$$P_b = 2\cdot Q\left(\sqrt{2\cdot\gamma_b}\right)$$ (again, I wish I had labeled differently, and eliminated the factor of

Let's compare with a detector that doesn't exploit memory, to see if the Viterbi effort is worthwhile. The constellation is shown below, with the logic level shown in parentheses by each point.



This biorthogonal set is familiar. A union bound on BER is good enough, for reasons you investigated in a previous assignment, so it's

$$P_b = 2\cdot Q\left(\sqrt{\gamma_b}\right)$$

which is 3 dB worse than the Viterbi detector. Yay!

This one could also benefit from the same relabeling as in Viterbi. Just make $-\mathbf{s}_1$ and $\mathbf{s}_2$ carry a logic 1, and $\mathbf{s}_1$ and $-\mathbf{s}_2$ carry a logic 0. The factor of 2 in the BER is then eliminated.