SIMON FRASER UNIVERSITY
School of Engineering Science

ENSC 428 Data Communications

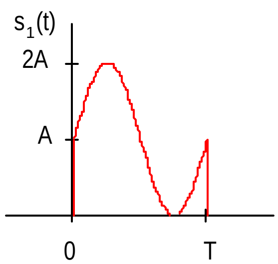
Solutions to Midterm Exam **Semester 01-1**

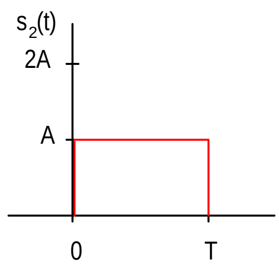
1. For convenience, define the rectangular pulse

$$\mathrm{rect}(t) := \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The two pulse shapes are then

$$s_1(t) = A\cdot \mathrm{rect}\left(\frac{t}{T}\right) + A\cdot \sin\left(2\cdot\pi\cdot\frac{t}{T}\right) \qquad s_2(t) = A\cdot \mathrm{rect}\left(\frac{t}{T}\right)$$





(a, 3 marks). The simplest of the many possible non-orthogonal bases of the space is just $s_1(t)$ and $s_2(t)$ themselves.

(b, 3 marks). From observation, the two terms comprising $s_1(t)$ are orthogonal and one of them is $s_2(t)$. A direct route to an orthonormal basis is to normalize the components. This gives

$$\psi_1(t) := \sqrt{\frac{2}{T}}\cdot \sin\left(2\cdot\pi\cdot\frac{t}{T}\right) \qquad \psi_2(t) := \frac{1}{\sqrt{T}}\cdot \mathrm{rect}\left(\frac{t}{T}\right)$$

since $$\int_0^T \sin\left(2\cdot\pi\cdot\frac{t}{T}\right)^2 dt = \frac{1}{2}$$ (sketch it and see)

A more difficult way to get an orthonormal basis is to apply the Gram-Schmidt procedure. Here it's best to start with $s_2(t)$. Starting with $s_1(t)$ is too horrible to contemplate, although it eventually gives a different, and equally correct, result So, normalizing $s_2(t)$ gives

$$\psi_2(t) := \frac{1}{\sqrt{T}} \cdot \mathrm{rect}\left(\frac{t}{T}\right)$$

Project $s_1(t)$ onto the $s_2(t)$ subspace to get the approximation

$$s_{1hat}(t) = \psi_2(t) \cdot \int_0^T s_1(t) \cdot \psi_2(t)\, dt$$

$$= \frac{1}{\sqrt{T}} \cdot \mathrm{rect}\left(\frac{t}{T}\right) \cdot \left(A \cdot \sqrt{T}\right) = A \cdot \mathrm{rect}\left(\frac{t}{T}\right)$$

This leaves the error

$$e_1(t) = s_1(t) - s_{1hat}(t) = A \cdot \sin\left(2 \cdot \pi \cdot \frac{t}{T}\right) \qquad \text{for} \qquad 0 \le t \le T$$

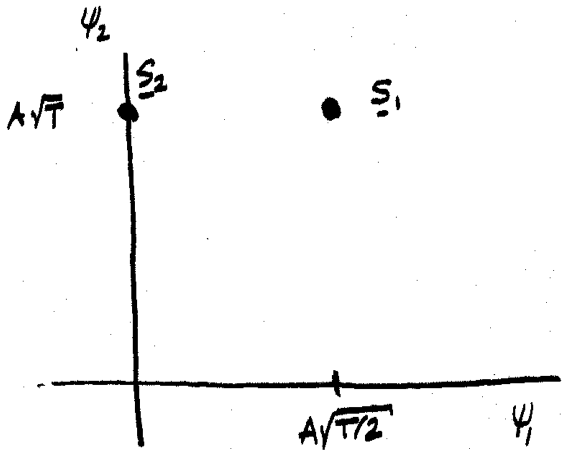
Normalization gives the next orthonormal basis function

$$\psi_1(t) = \frac{e_1(t)}{\mathrm{norm}\left(e_1(t)\right)} = \sqrt{\frac{2}{T}} \cdot \sin\left(2 \cdot \pi \cdot \frac{t}{T}\right) \qquad \text{for} \qquad 0 \le t \le T$$

which are the same ones we obtained by inspection.

(c, 3 marks) The signal constellation is obtained by calculating the coefficients of the signals with respect to the orthonomal basis. The various possible orthormal bases are all related by rotation and/or reflection, so the same is true of the resulting constellations. The one below uses the basis from part (b).

$$s_1(t) = A \cdot \sqrt{\frac{T}{2}} \cdot \psi_1(t) + A \cdot \sqrt{T} \cdot \psi_2(t) \qquad s_2(t) = A \cdot \sqrt{T} \cdot \psi_2(t)$$



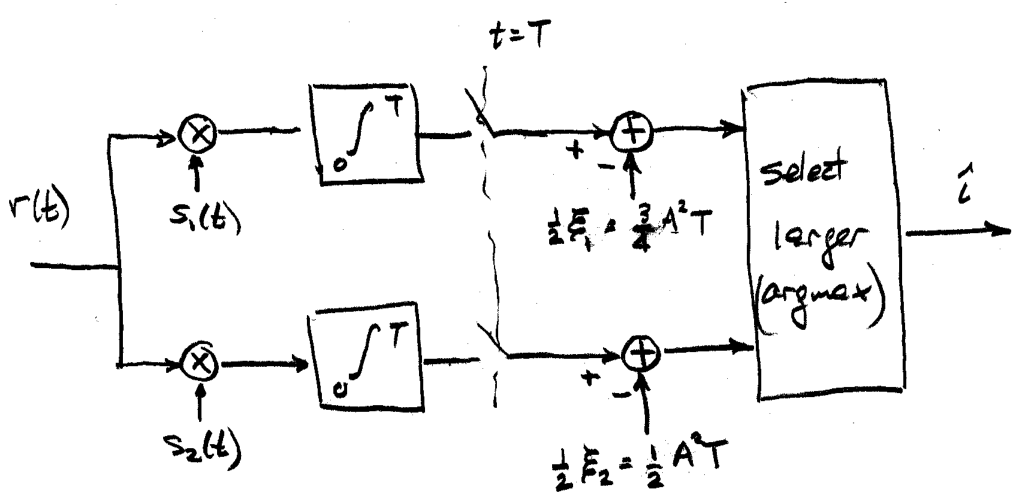
As a check, the energies of the two signals (squared lengths) are

$$E_1 = \frac{3}{2} \cdot A^2 \cdot T$$

$$E_2 = A^2 \cdot T$$

which agree with integrations of the signals $s_1(t)$ and $s_2(t)$.

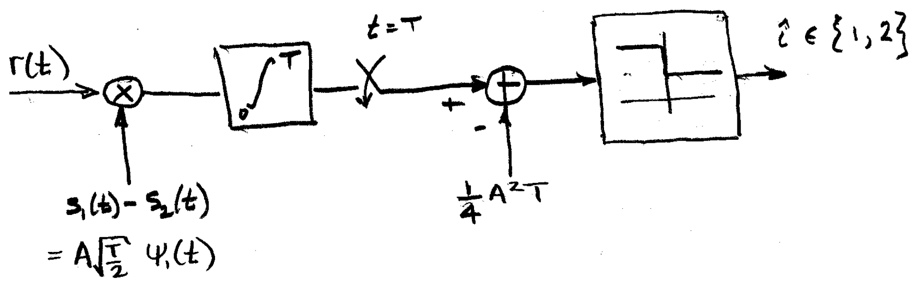
(d, 3 marks) Here's one possible two-correlator receiver



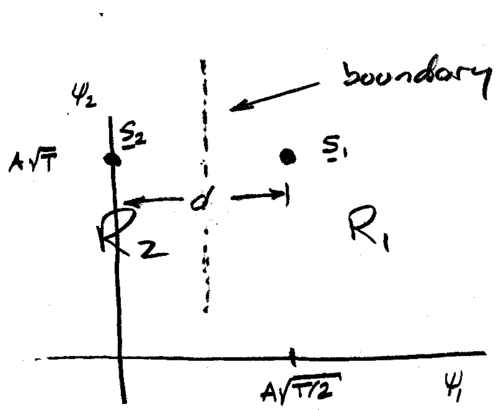
An equally satisfactory receiver correlates against the basis functions $\psi_1(t)$ and $\psi_2(t)$ to produce the components r_1 and r_1 of the received vector $\mathbf{r}$. The vector is delivered to a MAP receiver that does

$$\operatorname{argmax}_i \left(\mathbf{s_i}^T \cdot \mathbf{r} - \frac{E_i}{2} \right) \qquad \text{or (from the constellation)} \qquad \operatorname{sgn}\left(r_1 - \frac{A}{2} \cdot \sqrt{\frac{T}{2}} \right)$$

(e, 4 marks). Here's a one-correlator receiver. Like the two-correlator equivalent, an alternative correlates against a normalized version, which happens to be $\psi_2(t)$.



(f, 4 marks) With a decision boundary, the received space looks like this:



The ψ_2 dimension is irrelevant to the decision.

The signals are separated by $d = A\cdot\sqrt{\frac{T}{2}}$

The noise variance on each component is N_o, according to the question. It should have been $N_o/2$ - an unfortunate error on an exam sheet, so either choice was accepted in the grading. With $N_o/2$, the probability of bit error is

$$P_b = Q\left(\frac{\frac{d}{2}}{\text{noise\_std\_dev}}\right) = Q\left(\frac{d}{2}\cdot\sqrt{\frac{2}{N_o}}\right) = Q\left(\sqrt{\frac{A^2\cdot T}{2\cdot N_o}}\right)$$

2. (10 marks) Since the input is stationary, so is the ouput, and the sampling time is irrelevant. Simultaneity of the samples is important, because we want the correlation coefficient ρ. You can get at the variances and correlation coefficient in the time domain, through autocorrelation, or the frequency domain, through power spectrum.

In the time domain, follow the line set out in the section on Projection (notes 2.5.4 to 2.5.6). From this,

$$\sigma_1^2 = \frac{N_o}{2} \cdot \int_0^T s_1(t)^2 \, dt = \frac{N_o}{2} \cdot E_1 = \frac{No}{2} \cdot \frac{3}{2} \cdot A^2 \cdot T = \frac{3}{4} \cdot N_o \cdot A^2 \cdot T$$

$$\sigma_2^2 = \frac{N_o}{2} \cdot \int_0^T s_2(t)^2 \, dt = \frac{N_o}{2} \cdot E_2 = \frac{No}{2} \cdot A^2 \cdot T$$

$$\sigma_{12}^2 = \frac{N_o}{2} \cdot \int_0^T s_1(t) \, s_2 \cdot (t) \, 2 \, dt = \frac{N_o}{2} \cdot A^2 \cdot T$$

so that $$\rho = \frac{\sigma_{12}^2}{\sigma_1 \cdot \sigma_2} = \sqrt{\frac{2}{3}}$$

In the frequency domain, we can get the variances and covariance by

$$\sigma_1^2 = \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} \left(\left| S_1(f) \right| \right)^2 df \qquad \sigma_2^2 = \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} \left(\left| S_2(f) \right| \right)^2 df$$

$$\sigma_{12}^2 = \frac{N_o}{2} \cdot \int_{-\infty}^{\infty} S_1(f) \cdot \overline{S_2(f)} \, df$$

When you see painful integrals like this, you ask if Parseval can help you. He can. The time-domain equivalent of those frequency-domain inner products is much easier. In fact, it's just what we did further up the page. Of course, some other problem might be easier in the frequency domain.

3. (10 marks) Is MAP detection equivalent to use of the joint pdf in $\underset{i}{\text{argmax}}\ p_{\mathbf{r},\mathbf{s}}(\mathbf{r},\mathbf{s}_i)$?

Start with the MAP criterion. It is $\underset{i}{\text{argmax}}\ p_{\mathbf{s}_i|\mathbf{r}}(\mathbf{s}_i \mid \mathbf{r})$ not $\underset{i}{\text{argmax}}\ p_{\mathbf{r}|\mathbf{s}_i}(\mathbf{r} \mid \mathbf{s}_i)$

The latter is the ML criterion, suitable when the *a priori* distribution of $\mathbf{s}_i$ is uniform or unknown.

The MAP criterion is related to the joint pdf by

$$\underset{i}{\text{argmax}}\ p_{\mathbf{r},\mathbf{s}}(\mathbf{r},\mathbf{s}_i) = \underset{i}{\text{argmax}}\ p_{\mathbf{s}_i|\mathbf{r}}(\mathbf{s}_i \mid \mathbf{r})\, p_{\mathbf{r}}(\mathbf{r})$$

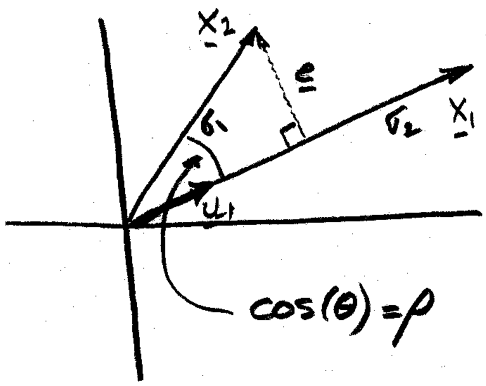
But $p_{\mathbf{r}}(\mathbf{r})$ is common to all the signal alternatives, so it makes no difference to the outcome.

Hence

$$\underset{i}{\text{argmax}}\ p_{\mathbf{r},\mathbf{s}}(\mathbf{r},\mathbf{s}_i) = \underset{i}{\text{argmax}}\ p_{\mathbf{s}_i|\mathbf{r}}(\mathbf{s}_i \mid \mathbf{r})$$

and maximization of the joint pdf is equivalent to a MAP decision. Of course, it is not equivalent to ML decisions, except if the prior distribution is uniform. Note also that the equivalence does not depend on a signal space or Gaussian noise - it is a general property of statistical decision making.

4. The vector diagram below summarizes the situation. The random variables have "lengths" equal to their standard deviations and an included angle θ that satisfies $\cos(\theta)=\rho$.



(a, 7 marks) The error is determined by the right triangle with hypotenuse of length $\sigma 2$. Therefore the "length" of the error is

$$\sigma_e = \sigma_2 \cdot \sin(\theta) = \sigma_2 \cdot \sqrt{1 - \cos(\theta)^2} = \sigma_2 \cdot \sqrt{1 - \rho^2}$$

and the mean squared error is

$$\sigma_e^2 = \sigma_2^2 \cdot (1 - \rho^2)$$

Another way of getting the same result is to go through the approximation step by step, rather like the G-S process. Start by normalizing X_1 to a unit vector

$$U_1 = \frac{X_1}{\sigma_1} \qquad \text{has unit standard deviation}$$

Then the MMSE approximation of X_2 is

$$X_{2hat} = (X_2, U_1) \cdot U_1 = \left\{ X_2, \frac{X_1}{\sigma_1} \right\} \cdot \frac{X_1}{\sigma_1} = \sigma_2 \cdot \cos(\theta) \cdot \frac{X_1}{\sigma_1} = \rho \cdot \frac{\sigma_2}{\sigma_1} \cdot X_1$$

with error

$$E_2 = X_2 - X_{2hat}$$

with variance

$$\sigma_e^2 = E(E_2) = E\left[(X_2 - X_{2hat})^2 \right] = E(X_2^2) - 2 \cdot E(X_2 \cdot X_{2hat}) + E(X_{2hat}^2)$$

$$= \sigma_2^2 - 2 \cdot \rho \cdot \frac{\sigma_2}{\sigma_1} \cdot \sigma_{21}^2 + \rho^2 \cdot \frac{\sigma_2^2}{\sigma_1^2} \cdot \sigma_1^2 = \sigma_2^2 - 2 \cdot \rho \cdot \frac{\sigma_2}{\sigma_1} \cdot \rho \cdot \sigma_1 \cdot \sigma_2 + \rho^2 \cdot \frac{\sigma_2^2}{\sigma_1^2} \cdot \sigma_1^2$$

$$= \sigma_2^2 \cdot (1 - \rho^2)$$

(b, 3 marks) Implicitly, we constrained the estimate to be linear, and minimized the MSE in that context. The calculations use only 2nd order statistics, not the underlying pdfs. Consequently, they apply to *any* random variables with finite variances. That is the answer I wanted. Not shown in class was the fact that the unconstrained estimate that minimizes MSE is the conditional mean. In general, this is a nonlinear function of X_1, but, if the variates *are* Gaussian, then the conditional mean is the linear MMSE estimate, so the procedure above is optimum.