2. REVIEW: PROBABILITY AND RANDOM PROCESSES

P+S Chapter 3

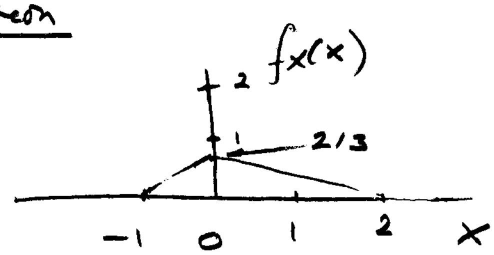
This section is intended as a review that highlights topics of importance to digital communications.

2.1 Probabilistics

P+S 3.1

Probability Density Function

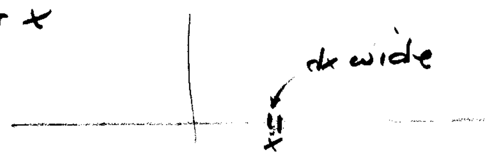
- An example:



  - notation

  $\rightarrow$ prob$(X = 1)$ = _____ ?

  - prob "X is in dx at x" is $f_X(x)\,dx$



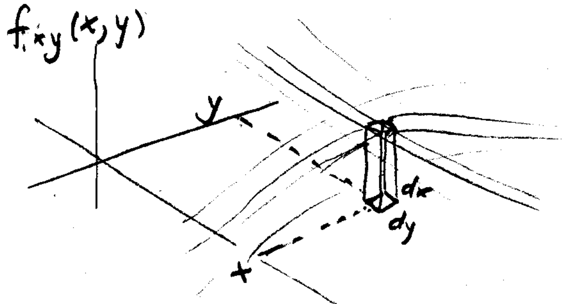
  - $\text{prob}(X \le 0) = \int_{-\infty}^{0} f_X(\xi)\,d\xi$

  $\text{prob}(X \le 0 \text{ or } 1 \le X \le 1.5) = \int_{-\infty}^{0} f_X(x)\,dx + \int_{1}^{1.5} f_X(\xi)\,d\xi$

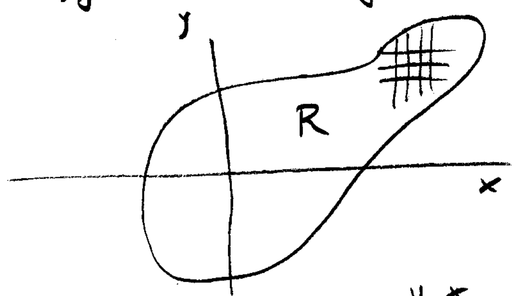
- Cumulative distribution function.

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \text{prob}(X \le x)$$

- Joint pdf $f_{xy}(x,y)\,dx\,dy$ is prob $\mathbf{x}$ in dx at x and $\mathbf{y}$ in dy at y

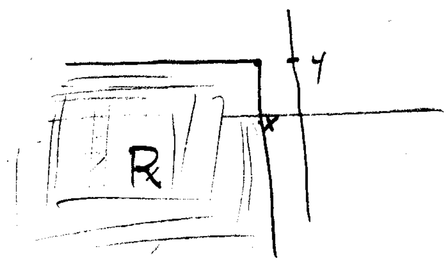


- prob x, y in some region R $= \iint_R f_{xy}(x,y)\,dx\,dy$



- joint cdf $F_{x,y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{xy}(\xi,\psi)\,d\xi\,d\psi$

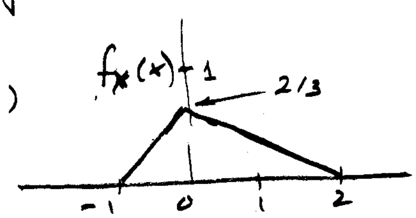
This is the prob that $\mathbf{x} < x$ and $\mathbf{y} < y$



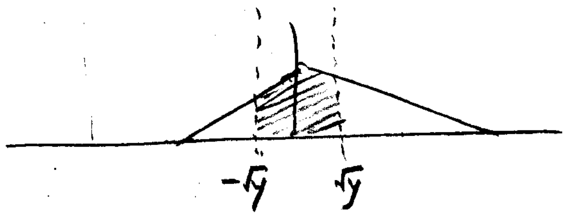
Transformations

- Suppose $\mathbf{y} = g(\mathbf{x})$. Then $\mathbf{y}$ is also a r.v. with a pdf. To determine $f_y(y)$ from $f_x(x)$ and $g(x)$ it is often easiest to go through the cdf.

<u>example</u> $g(x) = x^2$,



first get $F_y(y) = \text{prob}[y \le y] = \text{prob}\left[|x| \le \sqrt{y}\right]$



$$= F_x(\sqrt{y}) - F_x(-\sqrt{y})$$

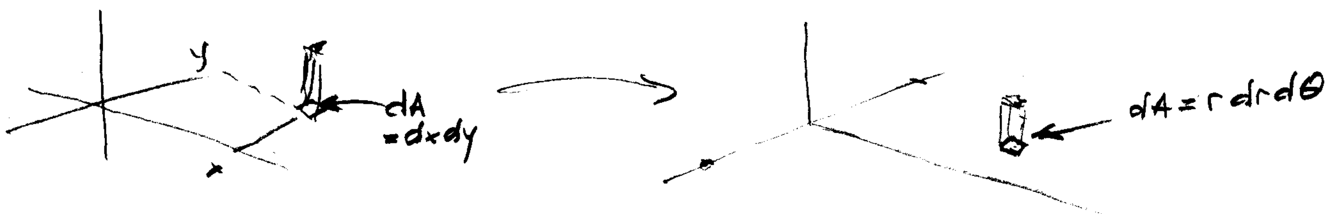
$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_x(\xi)\, d\xi$$

then $p_y(y) = \frac{d}{dy} F_y(y) = \frac{d}{dy} \int_{-y^{\frac{1}{2}}}^{y^{\frac{1}{2}}} f_x(\xi)\, d\xi$

$$= f_x(\sqrt{y})\, \tfrac{1}{2} y^{-\frac{1}{2}} - f_x(-\sqrt{y})\left(-\tfrac{1}{2} y^{-\frac{1}{2}}\right) = \frac{1}{2\sqrt{y}}\left(f_x(\sqrt{y}) + f_x(-\sqrt{y})\right)$$

Generally [P+S (3.1.10)] $f_y(y) = \sum_i \frac{f_x(x_i)}{|g'(x_i)|}$ where x_i are all the solutions of $g(x) = y$.

• Change of variables e.g. rect to polar

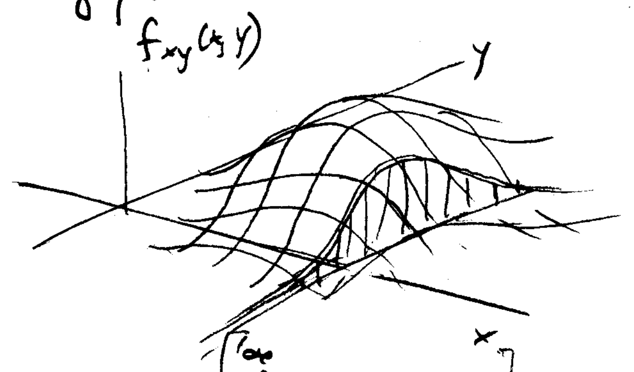


$$f_{xy}(\xi, \psi)\, dx\, dy \longrightarrow \underbrace{f_{xy}(r\cos\theta, r\sin\theta)\, r}_{f_{r\theta}(r,\theta)}\, dr\, d\theta$$

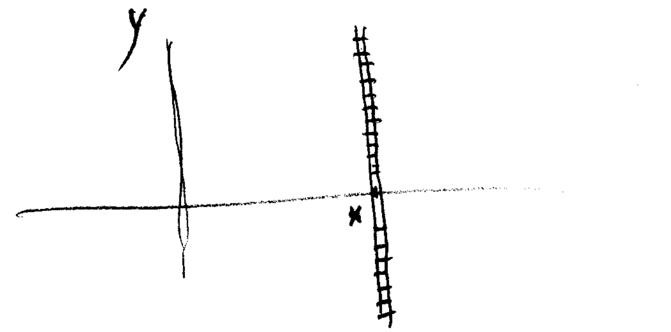
Don't forget the Jacobian!

Marginal pdf

- Suppose we have joint pdf $f_{xy}(x,y)$. How to determine the pdfs $f_x(x)$, $f_y(y)$ of the r.v.s observed singly?



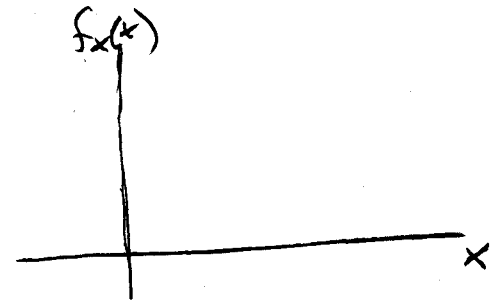
prob x in dx at x is sum over all cells that satisfy it



$$f_x(x)dx = \left[\int_{-\infty}^{\infty} f_{xy}(\xi,\psi)\,dy\right]dx$$

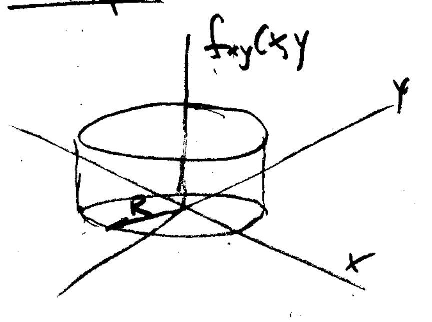
or

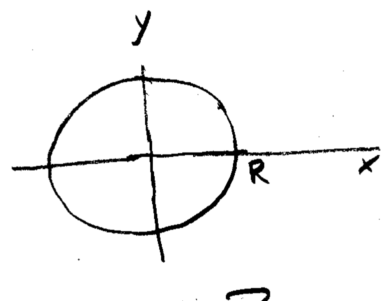
$$\boxed{\begin{aligned} f_x(x) &= \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dy \\ f_y(y) &= \int_{-\infty}^{\infty} f_{x,y}(x,y)\,dx \end{aligned}}$$



$$f_{uyz}(u,y,z) = \iint_{-\infty}^{\infty} f_{uvxyz}(u,v,x,y,z)\,dv\,dx$$

example





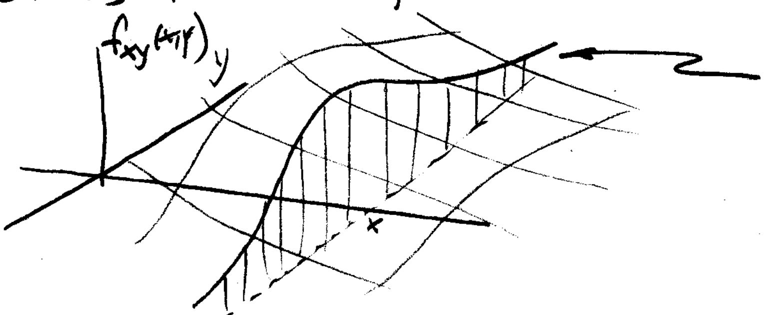
$$f_{xy}(x,y) = \begin{cases} \frac{1}{\pi R^2}, & x^2+y^2 \le R^2 \\ 0, & \text{elsewhere} \end{cases}$$

$f_x(x) =$?

- Can we go back again? Given $f_x(x)$ and $f_y(y)$, can we determine $f_{xy}(x,y)$?

Conditional pdf

- Sometimes the value of **x** affects the pdf of **y**.



This function of y (times dy) represents the relative probabilities that y lands in dy at y — but it does not integrate to 1.

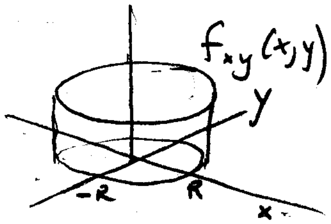
Solution: normalize it so it does integrate to 1. The conditional pdf

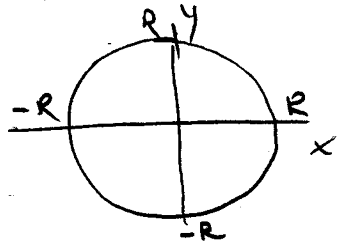
$$f_{y|x}(y|x) = \frac{f_{xy}(x,y)}{\int_{-\infty}^{\infty} f_{xy}(x,y)\,dy} = \frac{f_{xy}(x,y)}{f_x(x)}$$

Bayes' rule:

$$\boxed{f_{xy}(x,y) = f_{y|x}(y|x)\,f_x(x) = f_{x|y}(x|y)\,f_y(y)}$$

- example



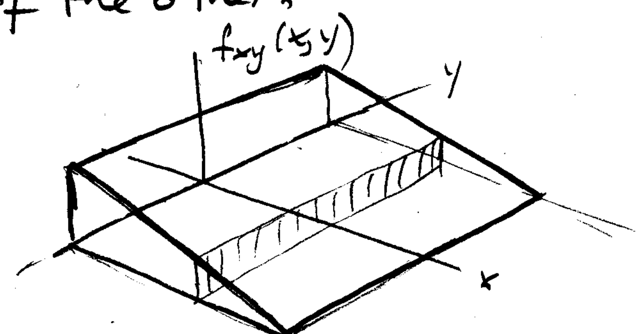


$$f_{x,y}(x,y) = \begin{cases} \frac{1}{\pi R^2}, & x^2+y^2 \le R^2 \\ 0, & \text{elsewhere} \end{cases}$$

what is $f_{y|x}(y|x)$?

- Independence. Sometimes the value of one variable has no influence on the pdf of the other:

$$f_{y|x}(y|x) = f_y(y)$$



The r.v.s x and y are "statistically independent."

- The joint pdf becomes

$$f_{xy}(x,y) = f_{y|x}(y|x)\,f_x(x) = f_y(y)\,f_x(x) \quad \text{a product}$$

This also shows that

$$f_{x|y}(x|y) = \frac{f_{xy}(x,y)}{f_y(y)} = \frac{f_y(y)\,f_x(x)}{f_y(y)} = f_x(x)$$

– Does it go the other way, too? If $f_{xy}(x,y) = f_x(x)\, f_y(y)$ are they independent?

- Recursive use of Bayes' rule. Several rv's w, x, y, z.

$$f_{wxyz}(w,x,y,z) = f_{w|x,yz}(w|x,y,z)\, f(x,y,z)$$
$$= f_{w|xyz}(w|x,y,z)\, f_{x|yz}(x|y,z)\, f_{y|z}(y|z)\, f(z)$$

Logarithm converts to sum:

$$\ln(f_{wxyz}(w,x,y,z)) = \ln(f_{w|xyz}(w|x,yz)) + \ln(f_{x|yz}(x|y,z)) + \ln(f_{y|z}(y|z)) + \ln(f(z))$$

Moments

- Expected values of powers of the r.v.s are moments

mean: $m_x = \int_{-\infty}^{\infty} x f_x(x)\, dx$ second moment: $\int_{-\infty}^{\infty} x^2 f_x(x)\, dx$

variance (2^{nd} central moment):

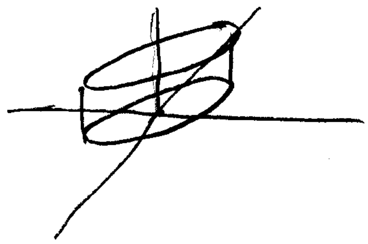
$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - m_x)^2 f_x(x)\, dx$$

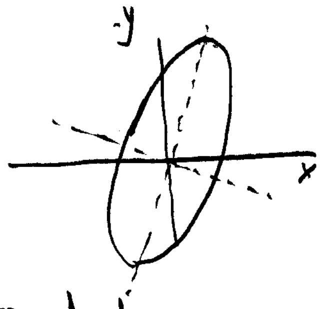
- For joint pdfs f_{xy}

$$\begin{tikzcd} & f_y(y) \arrow[r] \arrow[r, bend left] & m_y \quad \sigma_y^2 \\ f_{xy} \arrow[ur, "\int_x"] \arrow[dr, "\int_y"'] & & \\ & f_x(x) \arrow[r] \arrow[r, bend right] & m_x \quad \sigma_x^2 \end{tikzcd}$$

conditional mean $m_{y|x} = \int_{-\infty}^{\infty} y f_{y|x}(y|x)\,dy$

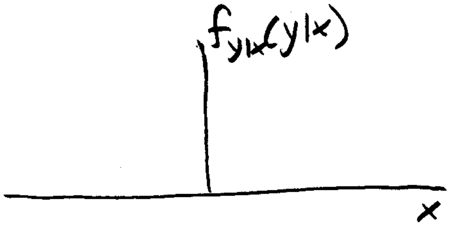
example

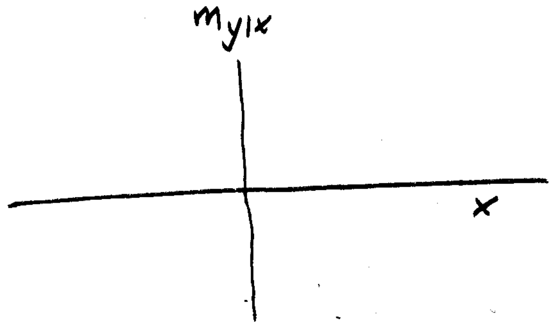




$$f_{xy}(x,y) = \begin{cases} \frac{1}{\pi r_1 r_2} & \text{inside} \\ 0 & \text{outside} \end{cases}$$

here $m_x = 0$ $m_y = 0$ but





- covariance: $\sigma_{xy}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-m_x)(y-m_y)\, f_{xy}(x,y)\,dx\,dx$

measures whether and how much they vary together.

what if they are independent?

correlation coefficient – normalized version

$$\rho = \frac{\sigma_{xy}^2}{\sigma_x \sigma_y}$$

- conditional variance $\sigma_{x|y}^2 = \int_{-\infty}^{\infty} (x - m_{x|y})^2\, p_{x|y}(x|y)\,dx$

Random vectors

- For multivariate pdfs, it is sometimes useful to group the rvs into a random vector

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad f_{\underline{x}}(\underline{x}), \quad f_{x_i}(x) = \underbrace{\iint \cdots \int}_{\text{all but } x_i} f_{\underline{x}}(\underline{x})\, d\underline{x}$$

$$\underline{m} = \begin{bmatrix} m_1 \\ \vdots \\ m_N \end{bmatrix} \quad \text{where} \quad m_i =$$

covariance matrix

$$C = E\left[(\underline{x}-\underline{m})(\underline{x}-\underline{m})^T\right] = \overline{\begin{bmatrix} x_1 - m_1 \\ x_2 - m_2 \\ \vdots \\ x_N - m_N \end{bmatrix} [x_1 - m_1, \; y_2 - m_2, \; \cdots \; x_N - m_N]}$$

$$= \begin{bmatrix} (x_1-m_1)^2 & (x_1-m_1)(x_2-m_2) & & (x_1-m_1)(x_N-m_N) \\ (x_2-m_2)(x_1-m_1) & (x_2-m_2)^2 & & \\ & & \ddots & \\ (x_N-m_N)(x_1-m_1) & & & (x_N-m_N)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & & \sigma_{1N}^2 \\ \sigma_{21}^2 & \sigma_2^2 & & \\ & & \ddots & \\ \sigma_{N1}^2 & & & \sigma_N^2 \end{bmatrix}$$

real, symmetric
positive semi definite

2.1.10

- Uncorrelated? Independent?
  - Random variables are termed uncorrelated if $\sigma_{xy}^2 = 0$
  - If x, y independent, then they are uncorrelated:

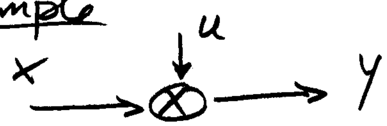
$$\sigma_{xy}^2 = \overline{(x - m_x)(y - m_y)} = \iint (x - m_x)(y - m_y)\, P_{xy}(x, y)\, dx\, dy$$

$$=$$

$$=$$

  - Doesn't work the other way — r.v.s can be uncorrelated, but statistically independent.

<u>example</u>



Here u, x are independent and zero mean.

y depends on x (if a sample of x is larger than usual, then y tends to have large magnitude than usual) but

$$\sigma_{xy}^2 = \overline{xy} = \overline{x^2 u} = \overline{x^2}\ \overline{u} = 0 \quad \text{since } \overline{u} = 0$$