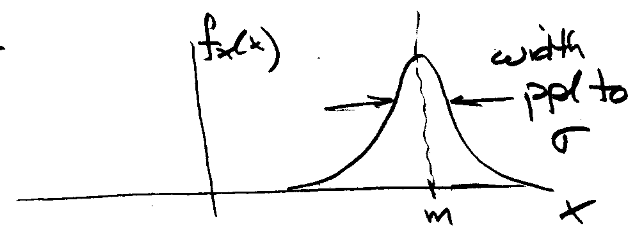
2.2 Gaussian pdf

- A pdf of enormous importance
  - characterized completely by 1st & 2nd order moments
  - central limit theorem: sums of identically distributed rvs have a pdf that looks increasingly Gaussian as number increases
    [A. Papoulis, Probability, Random Variables and Stochastic Processes, McGraw Hill 1984]
  - maximum entropy
  - many, many other properties

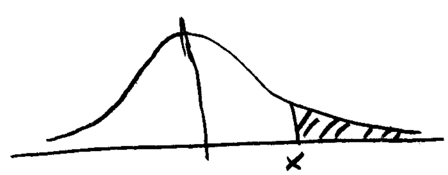
- scalar $\quad f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$



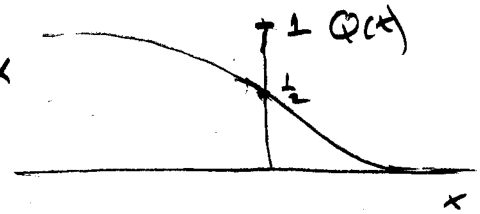
Aside: note $\displaystyle\int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}\left(\frac{x-m}{\sigma}\right)^2\right)dx = \sqrt{2\pi}\,\sigma$ irresp. of m

Area under the tail of zero mean, unit var Gaussian:

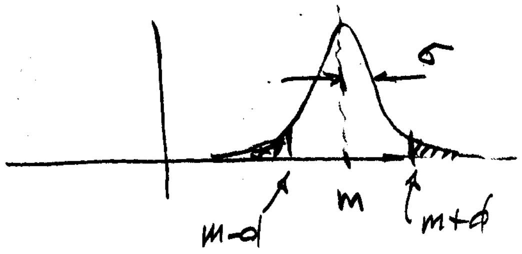
$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$Q(x) = 1 - F_x(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\alpha^2/2} d\alpha$$



Hence these areas



are each

$$Q\left(\frac{d}{\sigma}\right)$$

- bivariate Gaussian: suppose x, y have own variances and are not independent. From P+S p 158

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-m_x}{\sigma_x}\right)^2 + \left(\frac{y-m_y}{\sigma_y}\right)^2 \cdots\right.\right.$$

$$\left.\left. \cdots - 2\rho\left(\frac{x-m_x}{\sigma_x}\right)\left(\frac{y-m_y}{\sigma_y}\right)\right]\right\}$$

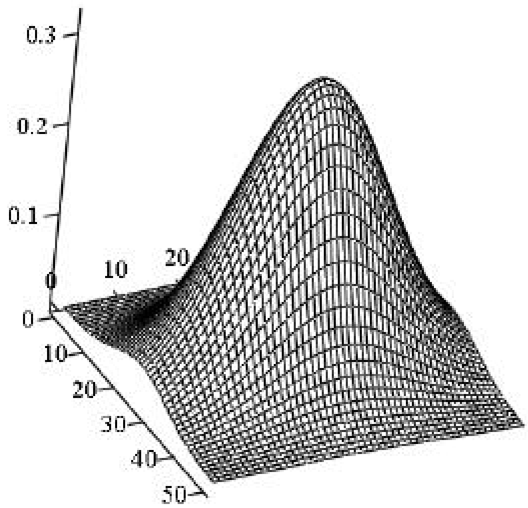
- multivariate Gaussian $\underline{x} = (x_1, x_2, \cdots x_N)^T$

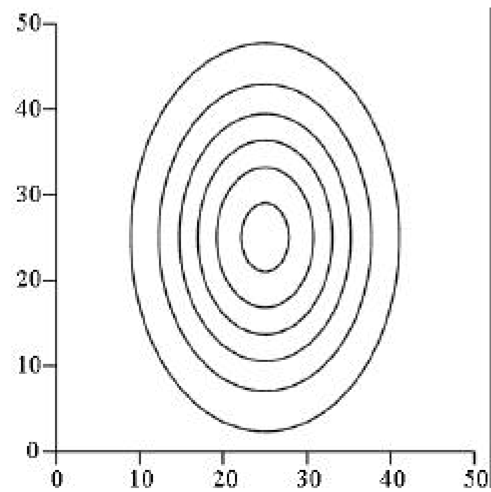
$$f_{\underline{x}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}} \exp\left\{-\frac{1}{2}(\underline{x}-\underline{m})^T C^{-1}(\underline{x}-\underline{m})\right\}$$

Is this the same as bivariate Gaussian for $N=2$?

$$C = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 \\ \sigma_{12}^2 & \sigma_2^2 \end{bmatrix} = \sigma_1\sigma_2 \begin{bmatrix} \sigma_1/\sigma_2 & \rho \\ \rho & \sigma_2/\sigma_1 \end{bmatrix}$$

Gaussian with covariance matrix $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ so $\rho = 0$





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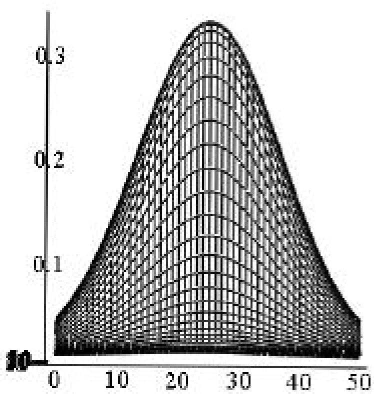
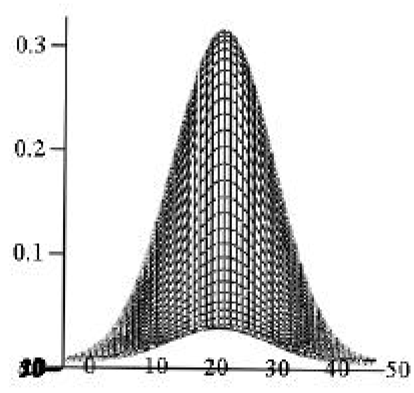
side views (parallel to axes):





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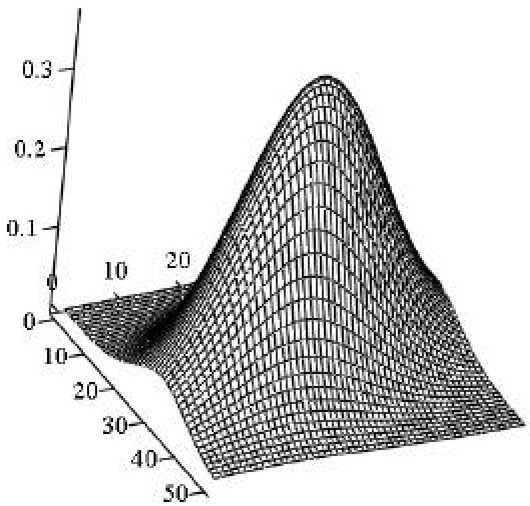
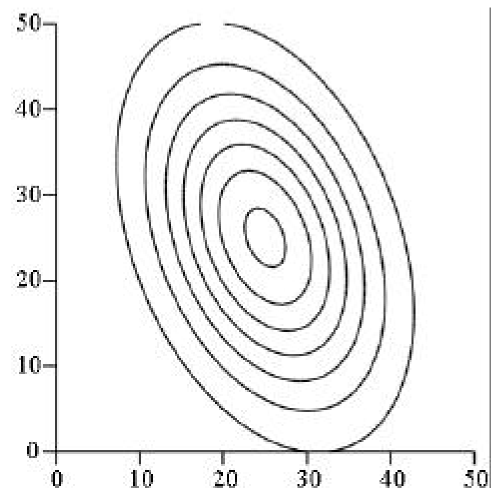
Gaussian with covariance matrix $\quad \mathbf{C} = \begin{bmatrix} 1 & 0.25 \\ 0.25 & 0.5 \end{bmatrix} \quad$ so $\quad \rho = 0.354$





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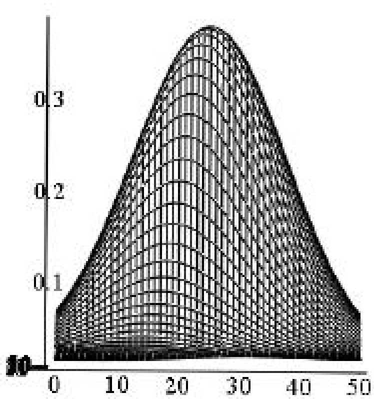

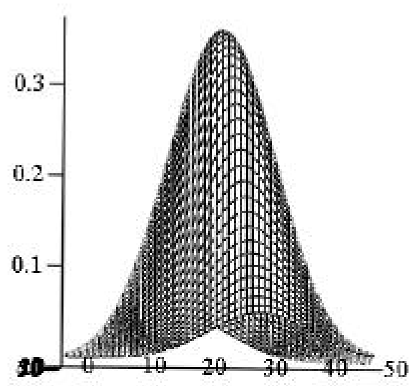

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- Occasionally, you need higher order moments of Gaussian r.v.s. If $x_1, x_2, \ldots x_N$ are jointly Gaussian, zeromean [JM Wozencraft and IM Jacobs, Principles of Communication Engineering John Wiley 1965, problem 3.13]

$$\overline{x_{i_1} x_{i_2} \cdots x_{i_L}} = \begin{cases} 0, & L \text{ odd} \\ \displaystyle\sum_{\substack{\text{all distinct} \\ \text{pairs of subscripts}}} \left[\sigma^2_{j_1 j_2} \sigma^2_{j_3 j_4} \cdots \sigma^2_{j_{L-1} j_L} \right], & L \text{ even} \end{cases}$$

examples:

$$\overline{x_1 x_2 x_3 x_4} = \sigma^2_{12} \sigma^2_{34} + \sigma^2_{13} \sigma^2_{24} + \sigma^2_{14} \sigma^2_{23}$$

$$\overline{x_1 x_2 x_3} = 0$$

If a variable is repeated, treat each occurrence as distinct

$$\overline{x_1^2 x_2 x_3} = \sigma^2_{11} \sigma^2_{23} + \sigma^2_{12} \sigma^2_{13} + \sigma^2_{13} \sigma^2_{12} = \sigma^2_{11} \sigma^2_{23} + 2\sigma^2_{12} \sigma^2_{13}$$

$$\overline{x_1^4} = 3\sigma_{11}^2 = 3\sigma_1^2$$

For complex Gaussian r.v.s, see

[W. McGee

- Linear transformations of Gaussian rv's are still Gaussian. If

$$f_{\underline{x}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}} \exp\left(-\frac{1}{2}(\underline{x}-\underline{m})^T C^{-1}(\underline{x}-\underline{m})\right)$$

and

$$\underline{v} = A\underline{x} + \underline{b}, \quad \underline{x} = A^{-1}(\underline{v}-\underline{b}), \quad d\underline{x} = \underbrace{|A^{-1}|}_{\text{Jacobian}} d\underline{v} = \frac{d\underline{v}}{|A|}$$

then $f_{\underline{x}}(\underline{x})\,d\underline{x}$ becomes

$$f_{\underline{v}}(\underline{v})\,dv = \frac{1}{\sqrt{(2\pi)^N |C|}} \exp\left(-\frac{1}{2}(A^{-1}(\underline{v}-\underline{b})-\underline{m})^T C^{-1}(A^{-1}(\underline{v}-\underline{b})-\underline{m})\right) \frac{d\underline{v}}{|A|}$$

$$= \frac{1}{\sqrt{(2\pi)^N |A|^2 |C|}} \exp\left\{-\frac{1}{2}(\underline{v}-\underline{b}-A\underline{m})^T A^{-T} C^{-1} A^{-1}(\underline{v}-\underline{b}-A\underline{m})\right\} d\underline{v}$$

$$= \frac{1}{\sqrt{(2\pi)^N |C_v|}} \exp\left\{-\frac{1}{2}(\underline{v}-\underline{m}_v)^T C_v^{-1}(\underline{v}-\underline{m}_v)\right\} d\underline{v}$$

where $\underline{m}_v = A\underline{m} + \underline{b}$, $C_v = ACA^T$