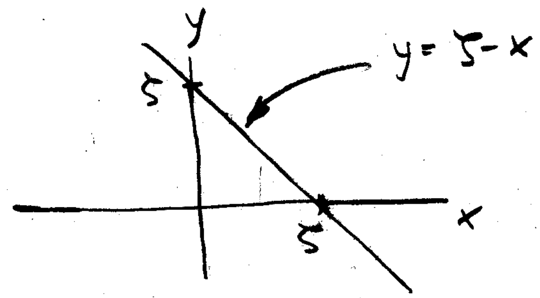
# 2.3 Sums of Random Variables

- Suppose $z = x + y$ and we have $p_{xy}(x, y)$. What is $p_z(\zeta)$?

  Easy answer through cdf:




$$F_z(\zeta) = \text{prob}[x + y \le \zeta]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\zeta - x} f_{x,y}(x, y)\, dy\, dx$$

then $f_z(\zeta) = \frac{d}{d\zeta} F_z(\zeta)$

- If $x, y$ independent (important case), then

$$F_z(\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\zeta - x} f_x(x)\, f_y(y)\, dx\, dy = \int_{-\infty}^{\infty} F_y(\zeta - x)\, f_x(x)\, dx$$

$$f_z(\zeta) = \frac{d}{d\zeta} F_z(\zeta) = \int_{-\infty}^{\infty} f_y(\zeta - x)\, f_x(x)\, dx$$

- The F transform of a pdf (the "characteristic function") simplifies it:

$$M_z(\omega) = \int_{-\infty}^{\infty} f_z(\zeta)\, e^{-j\omega\zeta}\, d\zeta$$

$$M_z(\omega) = M_x(\omega)\, M_y(\omega)$$

The characteristic function is sometimes called the "moment generating function"

$$\frac{d}{d\omega} M_x(\omega) = \int -jx f_x(x)\, e^{-j\omega x}\, dx \,, \quad M'(\omega)\Big|_{\omega=0} = -jm$$

$$\frac{d^2}{d\omega^2} M_x(\omega) = \int -x^2 f_x(x)\, e^{-j\omega x}\, dx \qquad M''(\omega)\Big|_{\omega=0} = -\overline{x^2}$$

$$\text{so} \quad \sigma^2 = -M''(0) + M'(0)^2$$

The characteristic function of a jointly Gaussian random vector $\underline{x}$ of length N has N frequency variables in $\underline{\omega}$

$$M_x(\underline{\omega}) = E\left[e^{-j\underline{\omega}^T \underline{x}}\right] = \int \cdots \int e^{-j(\omega_1 x_1 \cdots \omega_N x_N)} p_{x_1 \cdots x_N}(x_1, \cdots x_N)\, dx_1 \cdots dx_N$$

$$= \exp\left(-\tfrac{1}{2}\,\underline{\omega}^T C\, \underline{\omega}\right) \text{ for Gauss, zeromean cov } C$$

- The mean of a sum is the sum of the means.

  Always?

- What about the variance of a sum?

$$z = \sum_i x_i \qquad m_z = \sum_i m_i$$

$$\sigma_z^2 = \overline{(z-m_z)^2} = \overline{\left(\sum_i x_i - m_i\right)^2} = \overline{\sum_i \sum_j (x_i - m_i)(x_j - m_j)}$$

$$= \sum_i \sum_j \sigma_{ij}^2 = \sum_i \sigma_i^2 + 2 \sum_i \sum_{j \neq i} \sigma_{ij}^2$$

Aside: another way to get this is to pack $\{x_i\}$ into $\underline{x}$.

Then $z = \underline{e}^T \underline{x}$, where $\underline{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

and $z - m = \underline{e}^T (\underline{x} - \underline{m}_x)$

$$\overline{(z-m)^2} = \overline{\underline{e}^T (\underline{x} - \underline{m}_x)(\underline{x} - \underline{m}_x)^T \underline{e}} = \underline{e}^T C_x \underline{e}$$

$$= \sum_i \sum_j \sigma_{ij}^2$$

→ variance of a sum equals sum of variances iff r.v.s are uncorrelated; i.e. $\sigma_{ij}^2 = 0$ for $i \neq j$

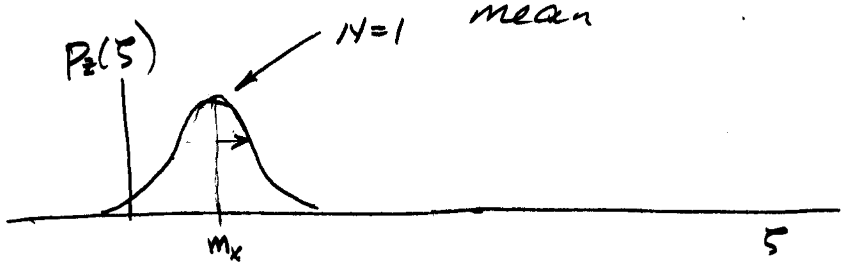
i.e. $C$ is diagonal

- Add (or average) $N$ i.i.d. variates
  - if $z = \sum_{i=1}^{N} x_i$

    then $m_z = N\, m_x \qquad \sigma_z^2 = N \sigma_x^2$

    but it's $\sigma$ that we compare with $m$, not $\sigma^2$

    $\sigma_z = \sqrt{N}\, \sigma_x$ scatter increases more slowly than mean




for $N = 4$ ?

Note – not necessarily Gaussian to start

  - coefficient of variation measures scatter relative to mean $C = \dfrac{\sigma}{m}$ . For $z$, $C_z =$

  - average: $z = \frac{1}{N} \sum_{i=1}^{N} x_i$

    so $m_z = m_x \qquad \sigma_z^2 = \frac{1}{N^2} N \sigma_x^2 = \frac{\sigma_x^2}{N} \qquad \sigma_z =$