2.4 Random Processes

$\boxed{\text{P+S } 3.2,\ 3.3}$

- Conceptually, the waveform you oberve $x(t)$ is a sample function drawn from an ensemble of possible waveforms. Analogous to sample values and ensemble for random variables.

- A general, but hard, way to specify a process is with a rule that gives the joint pdf of any set of samples

$$f_{x(t_1)\,x(t_2)\cdots x(t_N)}(x_1, x_2, \cdots x_N) \qquad \text{for any choice of } N$$

Stationarity

- A process is stationary if those joint pdfs depend only on the intersample spacing, not the start time. The stats don't change with time.

- Weaker condition: wide sense stationarity. A process is WSS if its first and second joint moments do not depend on time:

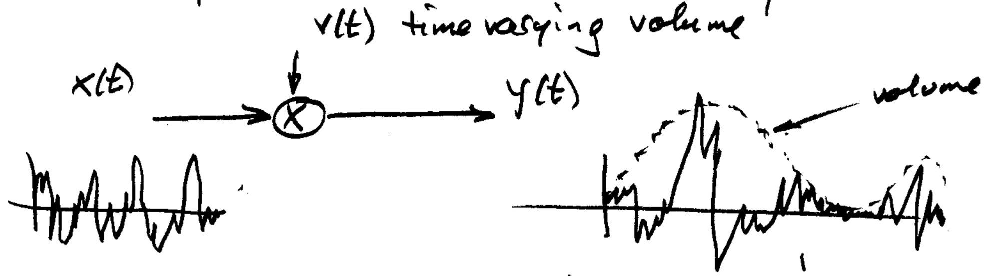
$$m_x(t) = E\left[x(t)\right] = \int_{-\infty}^{\infty} x\, f_{x(t)}(x)\,dx = \int_{-\infty}^{\infty} x\, f_x(x)\,dx = m_x$$

and

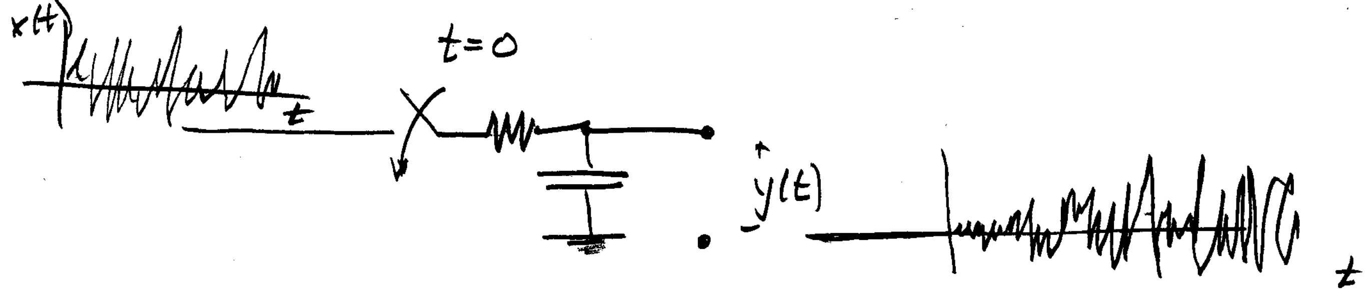
$$\sigma^2(t_1,t_2) = E\Big[\big(x(t_1)-m\big)\big(x(t_2)-m\big)\Big] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\alpha-m)(\beta-m)\, f_{x(t_1)x(t_2)}(\alpha,\beta)\,d\alpha\, d\beta$$

$$= \sigma^2(t_2-t_1)$$

- Not all processes are stationary:



another example — filter transients:



- Ensemble and time averages
  - In the calculations above, we used ensemble averages: average across all possible sample functions in the ensemble, weighted with their prob of occurrence. That's how we can get, for ex., $m(t)$ the average at a point in time

– In real life, we don't have access to the ensemble, so we make measurements and averages on the sample function at hand. Example:

Instead of $m = E[x(t)]$, use $m = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt$

or replace $E[x(t)\,x(t-\tau)]$ by $\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,x(t-\tau)\,dt$

This approach is useful primarily if the process is stationary.

– Do the two forms of average produce the same result?
  – If yes, then process is ergodic.
  – Read $\boxed{\text{P+S } 3.2.3}$

Autocorrelation Function and Power Spectrum

• Autocorrelation, cross correlation

$$R_x(t, t-\tau) = E[x(t)\,x(t-\tau)]$$

units?

$$R_{xy}(t, t-\tau) = E[x(t)\,y(t-\tau)]$$

For zero mean processes, these are autocovariance and cross covariance functions.

They are a measure of coupling, or dependence, of samples at t and $t-\tau$. Correlation coeff

$$\rho(t,\tau) = \frac{R(t, t-\tau)}{\sqrt{R(t,t)\, R(t-\tau, t-\tau)}}$$

Not the whole story, since only 2nd order.

- For WSS, $R(t, t-\tau) \rightarrow R(\tau)$

  $\sigma^2 = R(0)$ doesn't matter when you sample, variance is the same

  $$\rho(\tau) = \frac{R(\tau)}{R(0)}$$

  For stationary, zero mean, $\lim_{\tau \to \infty} R(\tau) = 0$

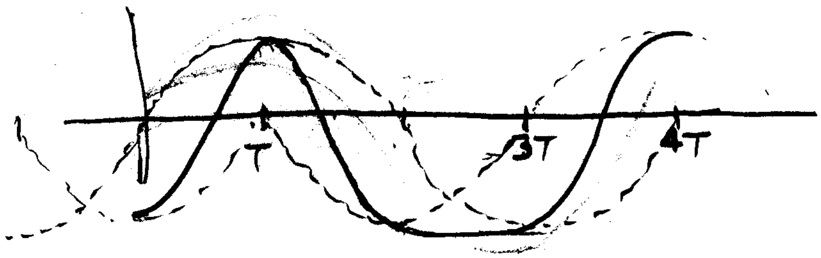
- Sometimes we have a mix of ensemble and time averages

  $$R_x(t, t-\tau) = E[x(t)\,x(t-\tau)]$$ may still depend on t

  where it makes sense, define (especially for cyclostationary)

  $$\bar{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(t, t-\tau)\, dt$$

• Data signals are often cyclostationary



$$x(t) = \sum_i a_i \, p(t - iT)$$

where the a_i data values are independent

This is a random process, defined by the data ensemble and the pulse shape. It is neither Gaussian nor WSS:

$$R_x(t, t-\tau) = E_{\{a\}}\left[x(t)\, x(t-\tau)\right] = \sum_i \sum_k \overline{a_i a_k}\, p(t-iT)\, p(t-\tau-kT)$$

$$= \sum_i \sum_k \delta_{ik}\, p(t-iT)\, p(t-\tau-kT)$$

$$= \sum_i p(t-iT)\, p(t-\tau-iT)$$

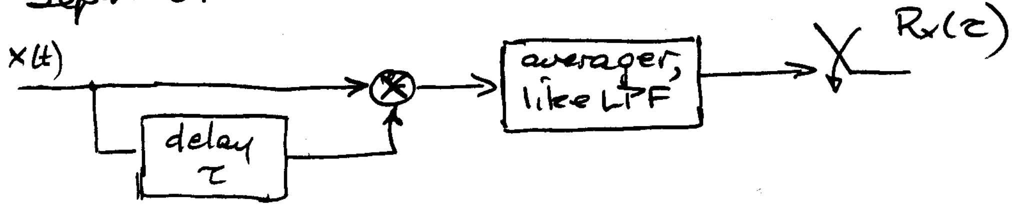
This is periodic in t (subst $t \leftarrow t+nT$ and reindex)

- power (variance) $R_x(t,t) = \sum_i p^2(t-iT)$ is periodically varying
- all stats, 2nd and higher, depend on sample spacing <u>and</u> t, but dependence on t is periodic

– If we want to ignore the periodicity, average over a cycle

$$\tilde{R}_x(\tau) = \frac{1}{T}\int_{-T/2}^{T/2}\left(\sum_i p(t-iT)\,p(t-\tau-iT)\right)dt$$

This gives the average of $x(t)x(t-\tau)$ over the ensemble and over t, keeping just the separation τ.



Integral becomes

$$\tilde{R}_x(\tau) = \frac{1}{T}\sum_i \int_{-T/2}^{T/2} p(t-iT)\,p(t-\tau-iT)\,dt$$

$$= \frac{1}{T}\sum_i \int_{(i-\frac{1}{2})T}^{(i+\frac{1}{2})T} p(u)\,p(u-\tau)\,du$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} p(t)\,p(t-\tau)\,dt$$

proportional to pulse autocorr f'n.

In particular,

$$\tilde{R}_x(0) = \frac{1}{T}\int_{-T/2}^{T/2} E_\theta[x^2(t)]\,dt$$ = average power of $x(t)$

$$= \underbrace{\frac{1}{T}}_{\text{pulse rate}}\cdot\underbrace{\int_{-\infty}^{\infty} p^2(t)\,dt}_{\mathcal{E}_p \text{ (energy per pulse)}}$$

Power Spectrum

$\boxed{\text{P+S 3.3}}$

- Intuitive definition in text. For any sample function $x(t)$, define $x_T(t)$ as a replica windowed to $|t| \leq T/2$ (not symbol period!). Then $X_T(f) \leftrightarrow x_T(t)$ is complex amplitude of component at f, $|X_T(f)|^2$ is energy spectral density:

$$\rightarrow \quad x_T(t) = \int_{-\infty}^{\infty} X_T(f)\, e^{j2\pi f t}\, df$$

$$\rightarrow \quad \int_{-T/2}^{T/2} x_T^2(t)\, dt = \text{energy} = \int_{-\infty}^{\infty} |X_T(f)|^2 df \quad \text{(Parseval)}$$

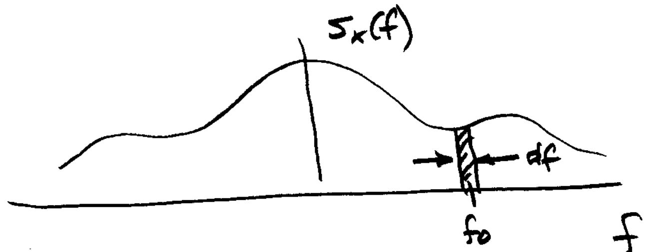
power spectral density $|X_T(f)|^2/T$:

$$\frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t) = \underset{\text{pwr over } T}{\text{average}} = \frac{1}{T}\int |X_T(f)|^2 df$$

ensemble average, then take limit to get power spectrum

$$S_x(f) = \lim_{T\to\infty} \frac{1}{T} E\left[|X_T(f)|^2\right]$$

It's a <u>density</u>: average (time, ensemble) power $= \int_{-\infty}^{\infty} S_x(f)\, df$

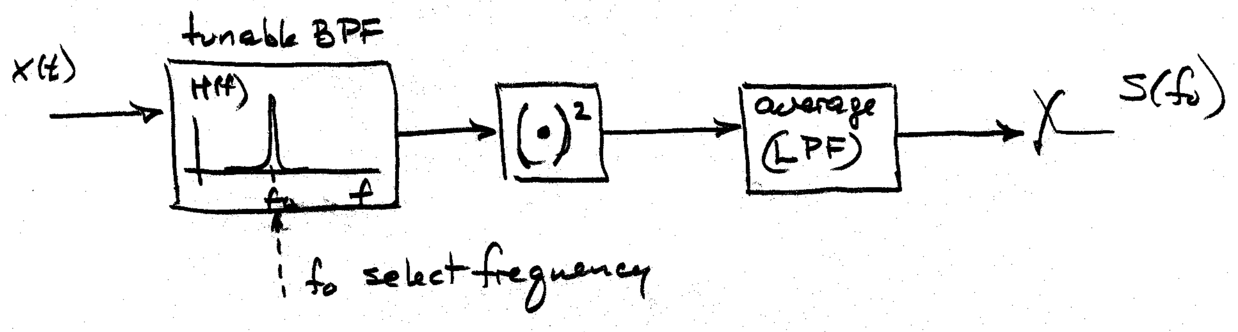


$S(f_0)$ is power in df at f_0

units?

symmetry? real? non-negative?

- What a spectrum analyzer measures:



- Wiener-Khinchin theorem.

Time averaged $\widetilde{R}_x(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(t, t-\tau)\, dt$

Power spectrum as defined above

$\widetilde{R}_x(\tau) \longleftrightarrow S_x(f)$

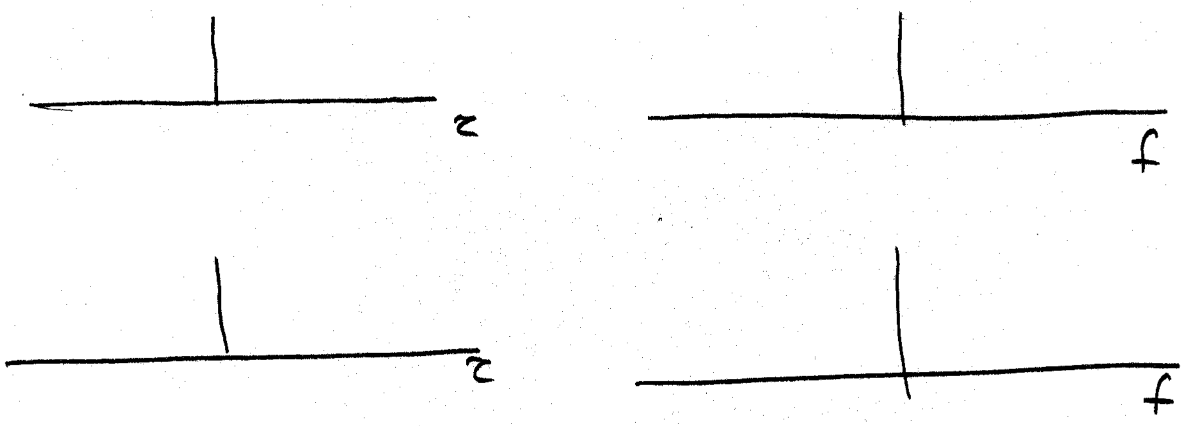
symmetries?

areas: $R_x(0) =$

$S_x(0) =$

units by consideration of transform.

- Extent in one domain $\propto \left(\begin{array}{c}\text{fine structure dimension} \\ \text{in the other}\end{array}\right)^{-1}$



- Cross correlation and cross power densities (for stationary)

$$R_{xy}(\tau) = E\left[x(t)\,y(t-\tau)\right]$$

symmetry?

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

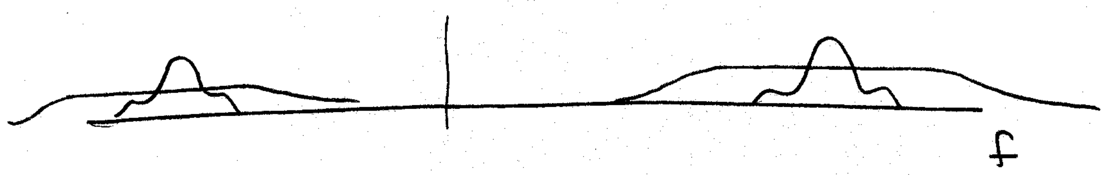
symmetry? real?
nonnegative?

(consider $y(t) = x(t-t_0)$)

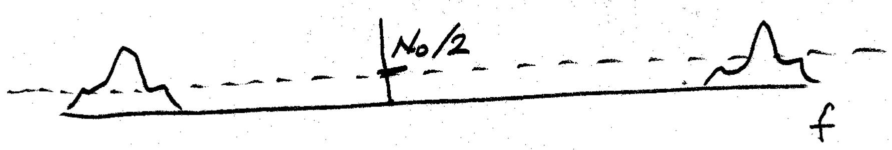
2.4.10

White Noise – A Convenient Fiction

- Often, noise power spectrum is flat across the signal band of interest

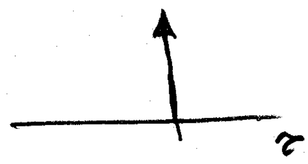


Simple model: flat everywhere, since makes no difference



$$S_n(f) = N_0/2, \quad \text{all } f$$

- autocorrelation function $R_n(\tau) = \mathcal{F}^{-1}[S_n(f)] = \frac{N_0}{2}\delta(\tau)$



- ugly bits:
  - infinite power
  - infinite variance for any sample
  - decorrelates instantly
- nice bits:
  - easy to specify
  - when filtered, it is (usually) well behaved, no longer white
  - easy calculation of effect of filters on white noise

- so it's used as an "analytical testbed" for comparing methods, even if true noise isn't exactly white.