# 2.5 Processes, Filters and Projections

- WSS (wide sense stationary) processes interact with linear filters in straightforward ways — largely because expectation and filtering are both linear operations

- Filtering a process.

$$v(t) \longrightarrow \boxed{h(t)} \longrightarrow x(t) = \int v(t-\alpha)\, h(\alpha)\, d\alpha$$

Time domain:

$$R_x(\tau) = E[x(t)\, x(t-\tau)] = E\left[\iint v(t-\alpha)\, v(t-\tau-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta\right]$$

$$= \iint R_v(\tau+\beta-\alpha)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta$$

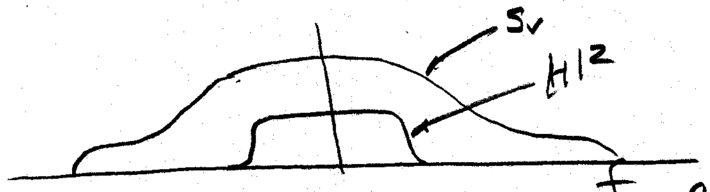
Freq domain:

$$R_x(\tau) = \iiint S_v(f)\, h(\alpha)\, h(\beta)\, e^{j2\pi f(\tau+\beta-\alpha)}\, df\, d\alpha\, d\beta$$

$$= \int S_v(f)\, H(f)\, H(-f)\, df$$

$$= \int S_v(f)\, |H(f)|^2\, df$$

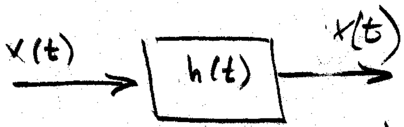
so $S_x(f) = S_v(f)\, |H(f)|^2$

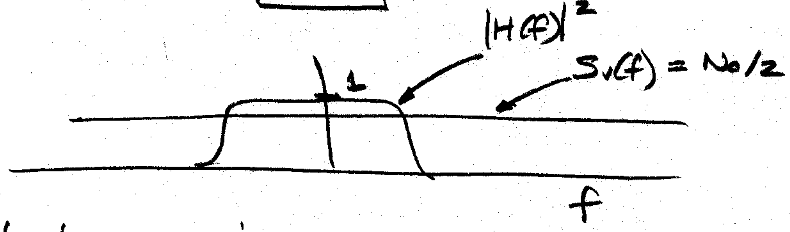



- $|H(f)|$ is an amplitude gain at freq f
- so $S_x(f)$ is still a power density
- phase doesn't matter — no effect on output power spectrum

- Noise bandwidth of a lowpass or bandpass.

Suppose we run a flat process thru a unit dc gain filter

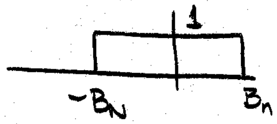



output power is

$$P_{out} = \int S_x(f)\,df = \frac{N_o}{2}\int |H(f)|^2\,df$$

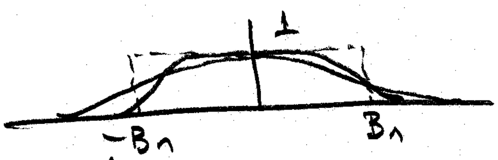
equivalent rectangular filter




gives $P_{out} = \frac{N_o}{2} \cdot 2B_n = N_o B_n$

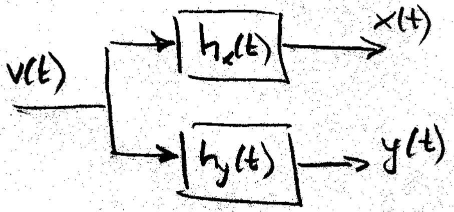
so noise bandwidth $B_n = \frac{1}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df = \int_0^{\infty} |H(f)|^2\,df$

Many filters can have the same $B_n$




and all have $P_{out} = N_o B_n$

- Cross correlation, cross power spectrum




If we can get cross stats between $x(t)$, $y(t)$, then we have them between input and output (allow one filter to be unit filter $h(t) = \delta(t)$)

- $$R_{xy}(\tau) = E\left[x(t)\,y(t-\tau)\right] = \iint \overline{v(t-\alpha)\,v(t-\tau-\beta)}\,h_x(\alpha)\,h_y(\beta)\,d\alpha\,d\beta$$

$$= \iint R_v(\tau+\beta-\alpha)\,h_x(\alpha)\,h_y(\beta)\,d\alpha\,d\beta$$

and

$$= \iiint S_v(f)\,h_x(\alpha)\,h_y(\beta)\,e^{j2\pi f(\tau+\beta-\alpha)}\,d\alpha\,d\beta\,df$$

$$= \int S_v(f)\,H_x(f)\,H_y(-f)\,df$$

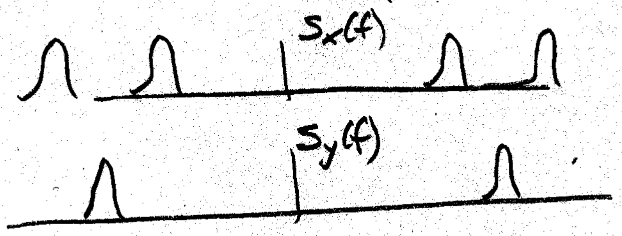
so $$S_{xy}(f) = S_v(f)\,H_x(f)\,H_y^*(f)$$

and $$R_{xv}(\tau) = \int R_v(\tau-\alpha)\,h_x(\alpha)\,d\alpha$$ input/output

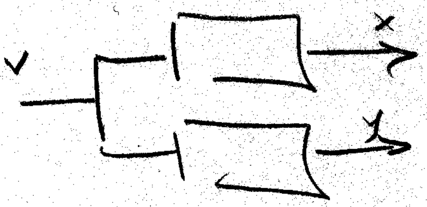
$$S_{xv}(f) = S_v(f)\,H_x(f)$$

A way to identify an unknown filter?

- Processes occupying different frequency bands are uncorrelated.




Imagine them to have been produced by different filters



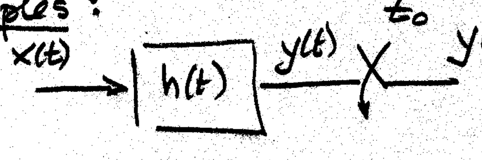

$$S_{xy}(f) = S_v(f)\, H_x(f)\, H_y^*(f)$$
$$\equiv 0$$
so $R_{xy}(\tau) \equiv 0$

## Projections

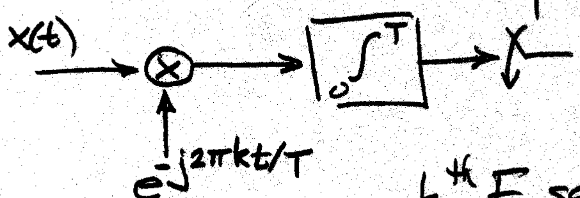
- In many applications, we calculate a correlation, or inner product, of a random process and a fixed function.

$$\xi = \int x(t)\, \varphi(t)\, dt \qquad \xi \text{ is a random variable}$$

examples:




$$y(t_0) = \int_{-\infty}^{t_0} x(\alpha)\, h(t_0-\alpha)\, d\alpha$$




$$X_k = \int_0^T x(t)\, e^{-j2\pi kt/T}\, dt$$

$k^{th}$ F series coeff of $x(t)$ in $[0, T]$.

These inner products, analogous to dot products, are often called projections, particularly if $\varphi(t)$ has unit energy $\int \varphi^2(t)\,dt = 1$.

- What are the stats of $\xi$?

$$\overline{\xi} = \int \overline{x(t)}\,\varphi(t)\,dt = \int m_x(t)\,\varphi(t)\,dt$$

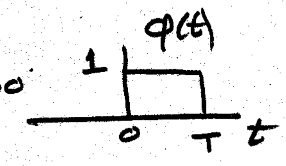
$$\overline{\xi^2} = \iint \overline{x(\alpha)\,x(\beta)}\,\varphi(\alpha)\varphi(\beta)\,d\alpha\,d\beta = \iint R_v(\alpha-\beta)\varphi(\alpha)\varphi(\beta)\,d\alpha\,d\beta$$

It's like a quadratic form in continuous time.
Recall sum of random variables.

If $R_v(\alpha-\beta) = \frac{N_0}{2}\delta(\alpha-\beta)$ then $\overline{\xi^2} = \frac{N_0}{2}\iint \delta(\alpha-\beta)\,\varphi(\alpha)\varphi(\beta)\,d\alpha\,d\beta$

$$= \frac{N_0}{2}\int \varphi^2(\alpha)\,d\alpha = \frac{N_0}{2}\,\mathcal{E}_\varphi$$

Another special case — just integrate $x(t)$, so




$$\overline{\xi^2} = \int_0^T\!\!\int_0^T R_v(\alpha-\beta)\,d\alpha\,d\beta$$

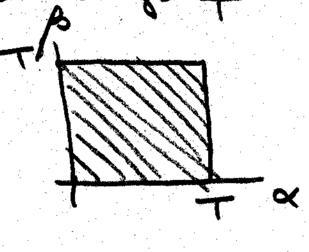
and if white noise $R_v(\tau) = \frac{N_0}{2}\delta(\tau)$

then

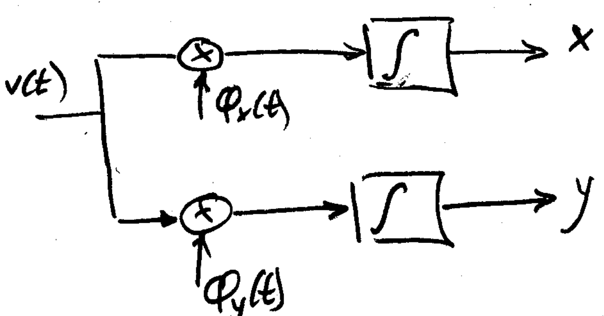
$$\overline{\xi^2} = \int_0^T \frac{N_0}{2}\,d\alpha = \frac{N_0}{2}T$$

2nd moment (var for zero mean) increases linearly with integration time

Aside: $\int_0^T\!\!\int_0^T R_v(\alpha-\beta)\,d\alpha\,d\beta$ is often easier with a change of variables



- If two correlatcons




The cross correlatcon of the two random variables is

$$\overline{xy} = \iint \overline{v(\alpha)\, v(\beta)}\, \varphi_x(\alpha)\varphi_y(\beta)\, d\alpha\, d\beta = \int R_v(\alpha-\beta)\varphi_x(\alpha)\varphi_x(\beta)\, d\alpha\, d\beta$$

If white noise $\overline{xy} = \frac{N_0}{2} \int \varphi_x(\alpha)\varphi_y(\alpha)\, d\alpha$ inner product of fixed waveforms

- Questcon: What are the stats of the Fourier series coefficients of white noise in $[0, T]$?

- Finally, freq domain equivalents for the sketch above:

$$\overline{x^2} = \iint R_v(\alpha-\beta)\, \varphi_x(\alpha)\varphi_x(\beta)\, d\alpha\, d\beta = \iiint S_v(f) e^{j2\pi f(\alpha-\beta)} \varphi_x(\alpha)\varphi_x(\beta)\, d\alpha\, d\beta$$

$$= \int S_v(f) |\Phi_x(f)|^2 df$$

$$\overline{xy} = \iint R_v(\alpha-\beta)\varphi_x(\alpha)\varphi_y(\beta)\, d\alpha\, d\beta = \int S_v(f)\, \Phi_x(f)\, \Phi_y^*(f)\, df$$