4. WHY DIGITAL?

- We have seen the advantages of digital format from the perspective of uniform treatment of a variety of digital and analog sources. Yet digital transmission of analog signals was taking over before uniformity was a consideration.

  Why?

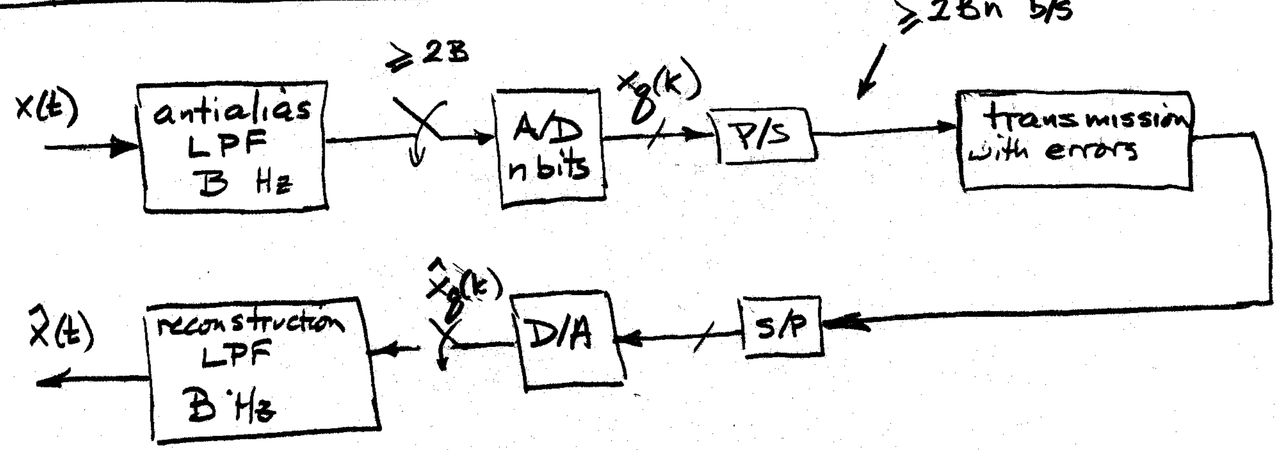
- In this section, we explore digital transmission of analog sources, including:
  - basic tradeoffs among

$$\begin{array}{ccc} \text{channel bandwidth} & \longleftrightarrow & \text{channel SNR} \\ & \nwarrow\!\!\searrow \quad \nearrow\!\!\swarrow & \\ & \text{output SNR} & \end{array}$$

  - comparison with FM on the same basis
  - ways to improve the performance and spectral efficiency of digital.

4.1 Quantization and PCM

Simple Model of PCM

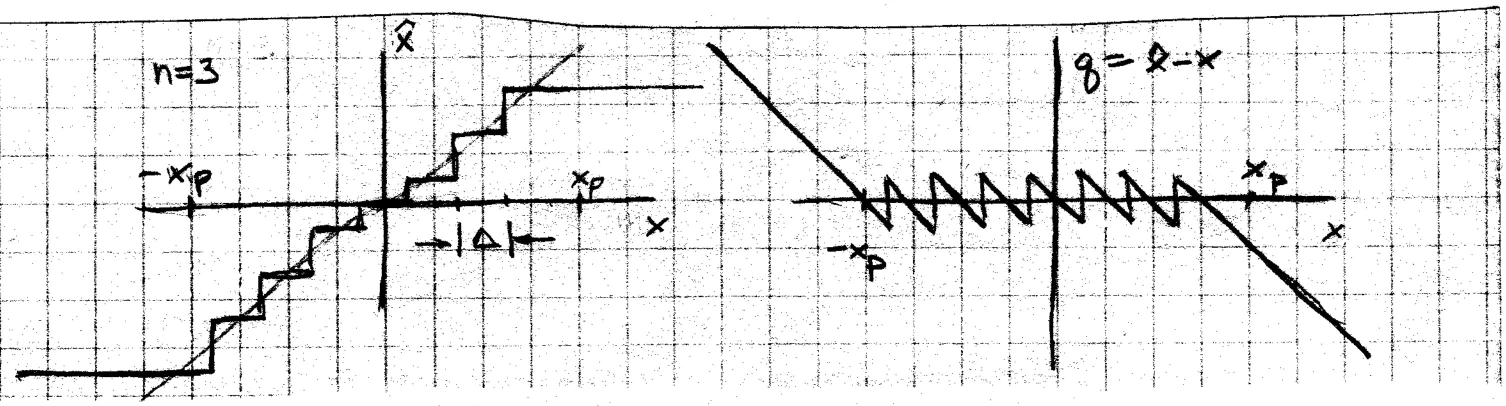


With an infinite number of quantization bits and no transmission errors, $\hat{x}(t) = x(t)$. (Also B big enough to accommodate $S_x(f)$.)

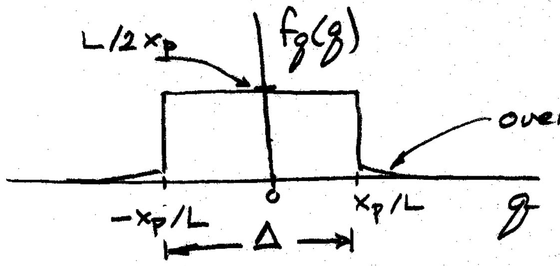
Quantization

- First, look at linear quantizers (A/D converters)

  n bits $\quad L = 2^n$ levels $\quad$ step size $\Delta = 2x_p 2^{-n}$



Assume enough levels that g is uniformly distributed:

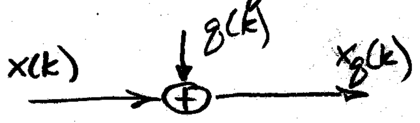


Choose input signal level to minimize quant + overload pwrs.

Typically "3σ loading" $\Rightarrow$ $x_p = 3\,\sigma_x$

This makes overload negligible, without too large a penalty in simple quantization noise.

- Model of the quantizer:



If enough levels, then $g(k)$ is:
- uncorrelated with $x(k)$
- white
- zero mean

Its variance (power) is then

$$\sigma_g^2 = \frac{1}{12}\left(\frac{2x_p}{L}\right)^2 = \frac{1}{3}\,x_p^2\,2^{-2n}$$

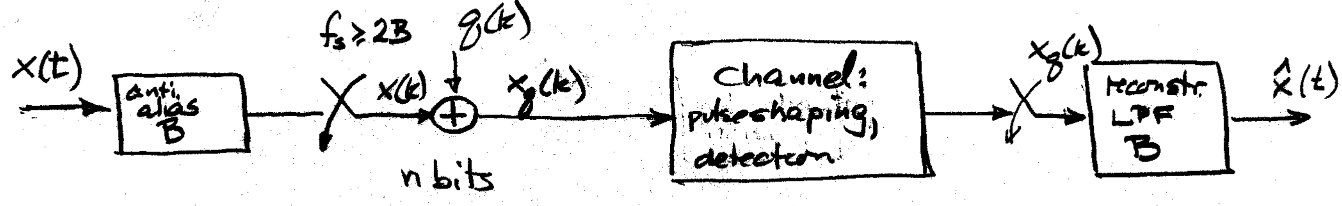
Add another bit ($n \leftarrow n+1$) and it drops by a factor $2^{-2} = 1/4$. "6 dB per bit".

Why does it scale as x_p^2? Does it matter?

$$\frac{\sigma_g^2}{\sigma_x^2} =$$

PCM With No Transmission Errors

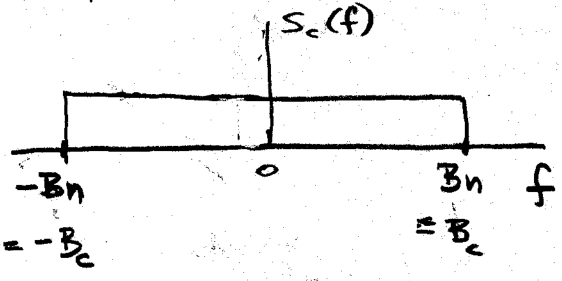
- Not realistic, so this model is temporary:



- Channel bandwidth occupied:
  - send $2Bn$ b/s
  - use sinc functions for the pulses in the channel:

$$p(t) = \operatorname{sinc}(2Bnt)$$ ← A poor choice:

so power spectrum in channel is

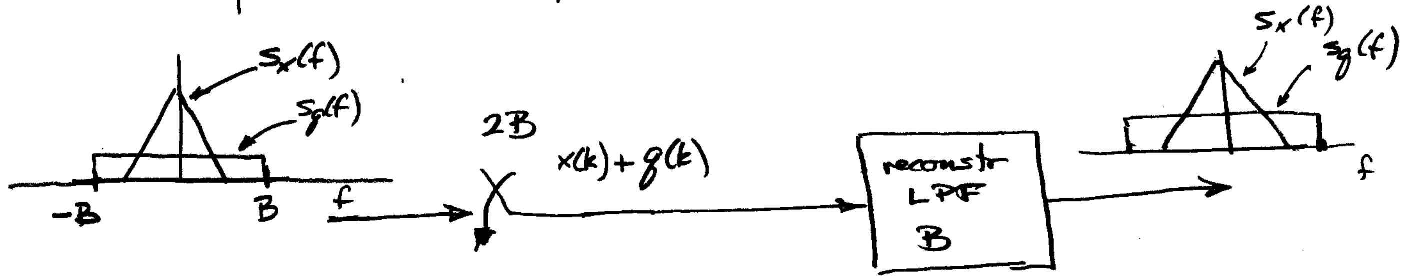


denote channel bandwidth B_c, b.w. expansion factor $\frac{B_c}{B} = n$

doubled when modulated if BPSK:

• Output SNR $SNR_o = \sigma_x^2/\sigma_q^2$

Justification: white quantization noise samples can be considered as samples from an input noise process that is flat over $[-B, B]$



So a sample of the output at any time has signal variance σ_x^2, quantization noise variance σ_q^2, and

$$SNR_o = \frac{\sigma_x^2}{\sigma_q^2} = \frac{\sigma_x^2\, 3}{x_p^2\, 2^{-2n}} = 3 \cdot 2^{2n} \underbrace{\left(\frac{\sigma_x}{x_p}\right)^2}_{\text{loading factor}} = \frac{1}{3}\, 2^{2n} \quad \left(\begin{array}{c} 3\sigma \\ \text{loading} \end{array}\right)$$

Looks as though output SNR increases rapidly with bandwidth expansion

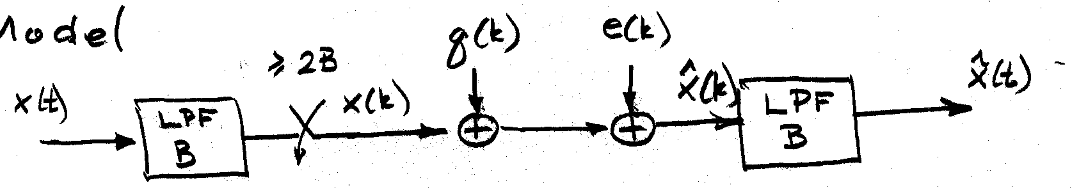
$$SNR_o = \frac{1}{3}\, 2^{2n} = \frac{1}{3}\, 2^{2(B_c/B)} = \frac{1}{3}\, e^{2\ln 2 (B_c/B)}$$

Seems impressive, compared to FM, in which $SNR_o \sim (B_c/B)^2$

<u>but</u> no transmission errors in the model. Naive!

PCM With Transmission Errors

• Model



Two independent sources of error. We understand $g(k)$, so now focus on $e(k)$, the effect of bit errors on linearly quantized signal.

• Transmission errors

4	3	2	1	0

for $n = 5$

An error in bit 0 results in an error in $\hat{x}(k)$ of $\pm$ one step:

$$e_0 = \begin{cases} \pm\Delta & \text{prob } P_e \\ 0 & \text{prob } 1-P_e \end{cases} \qquad \Delta = 2x_p 2^{-n}, \quad P_e = \text{BER}$$

(same for each bit)

An error in bit 1 is twice as large; in general,

$$e_i = \begin{cases} \pm \Delta 2^i, & \text{prob } P_e \\ 0 & \text{prob } 1-P_e \end{cases} \qquad \overline{e_i} = 0 \qquad e_i, e_k \text{ independent}$$

So transmission error signal at sample k is

$$e(k) = \sum_{i=0}^{n-1} e_i(k)$$

with variance

$$\sigma_e^2 = \sum_{i=0}^{n-1} \overline{e_i^2} = \Delta^2 P_e \sum_{i=0}^{n-1} 2^{2i} = 4x_p^2 2^{-2n} P_e \frac{2^{2n}-1}{2^2-1}$$

(why?)

$$= \frac{4}{3} x_p^2 \frac{2^{2n}-1}{2^{2n}} \approx \frac{4}{3} x_p^2 P_e$$

scales linearly with BER

- Output SNR and Threshold.

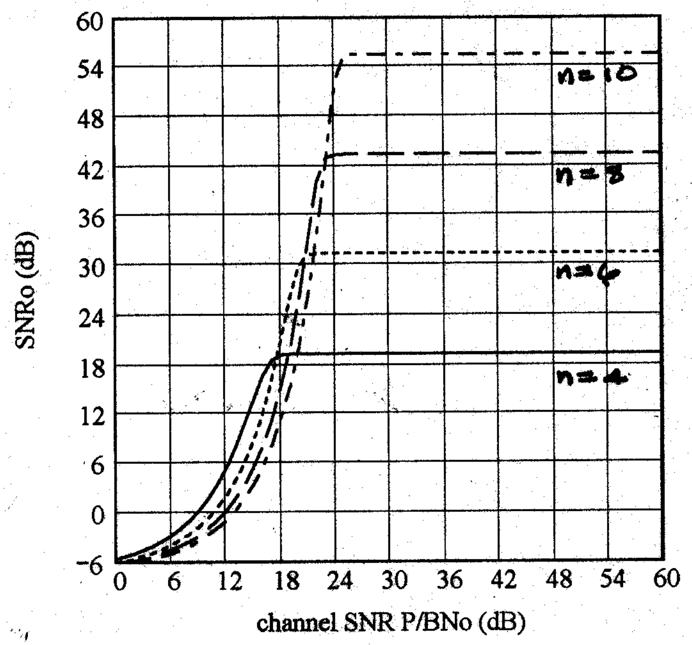
Total output error variance

$$\sigma_g^2 + \sigma_e^2 = \underbrace{\frac{1}{3} x_p^2\, 2^{-2n}}_{\text{dominates at low BER}} + \underbrace{\frac{4}{3} x_p^2\, \frac{2^{2n}-1}{2^{2n}} P_e}_{\text{dominates at high BER}}$$

and output SNR:

$$SNR_o = \frac{\sigma_x^2}{\sigma_g^2+\sigma_e^2} = \left(\frac{\sigma_x}{x_p}\right)^2 \frac{3}{2^{-2n} + 4\left(\frac{2^{2n}-1}{2^{2n}}\right)P_e}$$

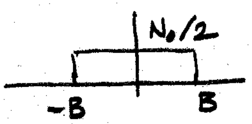
But P_e depends on channel SNR



$$P_e = Q\left(\sqrt{2E_b/N_o}\right)$$
$$= Q\left(\sqrt{2P/R_b N_o}\right) \quad P \text{ channel transmit power}$$
$$= Q\left(\sqrt{P/nBN_o}\right) \quad R_b = 2Bn$$
$$= Q\left(\sqrt{\Gamma/n}\right)$$

where Γ is an oddly defined channel SNR

$$\Gamma = \frac{P}{N_o B} = \frac{\text{transmit pwr}}{\text{noise power falling in signal band}}$$



useful for comparison with FM

For any Γ, there is an optimum number of bits/sample n:
- too few, and quantization error gets you
- too many, and bits don't have enough energy, so P_e is high and transmission errors get you.

Operate at threshold?