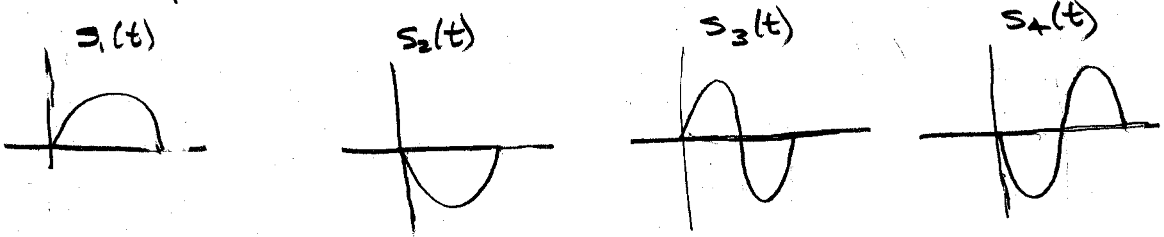
5.3 Detection in White Noise | P+S 7.2

Here we finally apply what we have learned so far to the basic problem of digital transmission – detection of isolated pulses in white noise.

Transmitted and Received Waveforms

- This is what we send: one of M possible waveforms, for example:

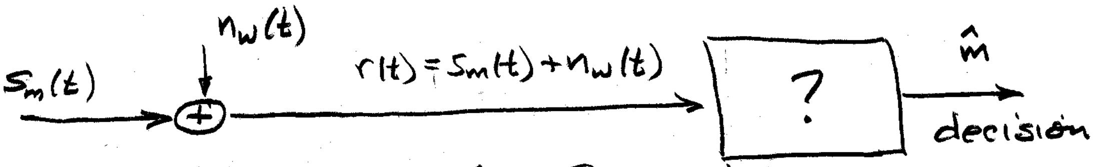


They have a priori probabilities $P(s_i)$ (a priori here means "before we look at the received signal")

Commonly: – M is a power of 2
– signals are equiprobable

but not always!

- This is what we receive:



white noise, PSD $N_0/2$, Gaussian

We make our decision about what signal was sent using the waveform $r(t)$.

AWGN =

Obtaining Sufficient Statistics

- Dealing with a complete waveform $r(t)$, $-\infty < t < \infty$ is awkward. We want to reduce it to a finite number of parameters without losing any information relevant to the decision. Here's how...

- Using <u>any</u> basis of the signal space, project $r(t)$ onto that basis. This represents the signal $s_m(t)$ exactly and the noise $n_w(t)$ approximately.

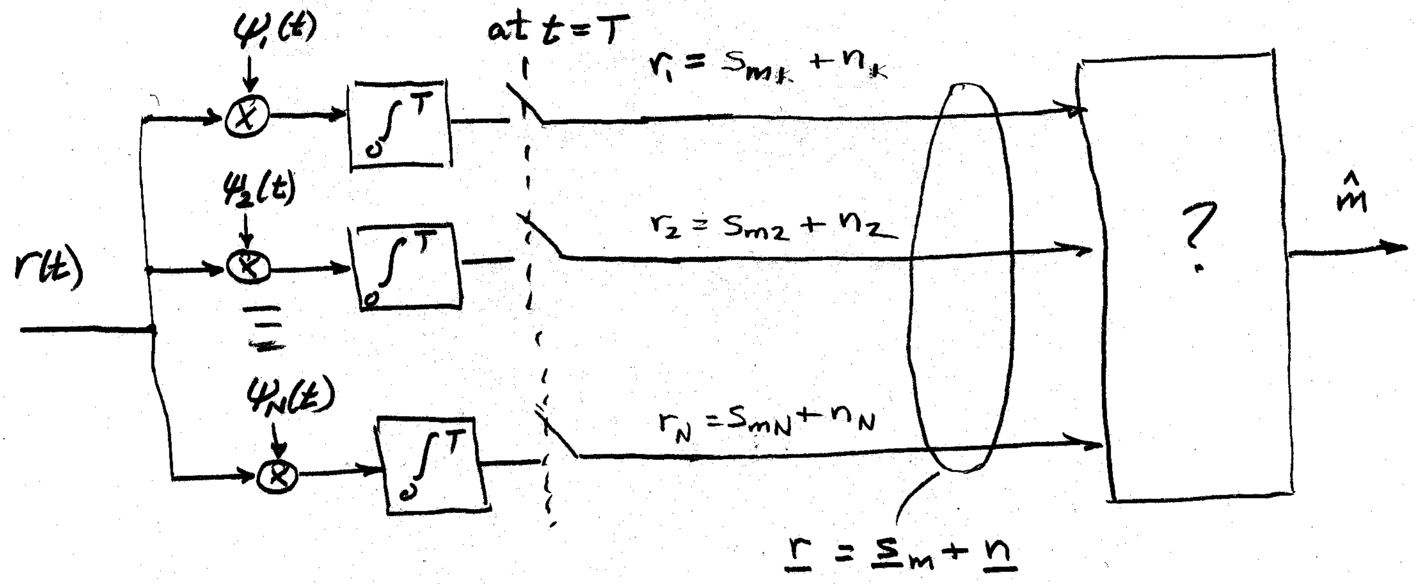
$$\hat{r}(t) = \hat{s}_m(t) + \hat{n}(t) = s_m(t) + \hat{n}(t)$$

so
$$r(t) = s_m(t) + \hat{n}(t) + e(t)$$

- "Theorem of Irrelevance"
  - Note $e(t)$ lies wholly in the orthogonal complement of the signal subspace. The basis functions of that complementary subspace are orthogonal to every function in the signal space, including $s_m(t)$, $\hat{n}(t)$.
  - Recall that white noise projections onto orthogonal functions are uncorrelated random variables, hence independent if Gaussian
  - Hence $e(t)$ gives no information about either $\hat{n}(t)$ or $s_m(t)$, so we don't need it.

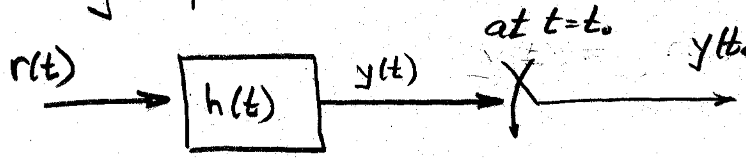
The projection $\hat{r}(t)$ is sufficient.

- Conceptually, it's simplest to use an orthonormal basis $\{\psi_i(t)\}$, $i = 1..N$. Obtaining the set of sufficient statistics becomes straightforward:



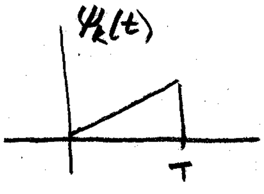
A set of correlators like this is the most common way to reduce an entire waveform to N numbers.

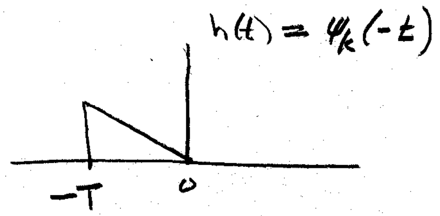
ASIDE
- An alternative implementation — the matched filter — is often more convenient for analysis and for analog implementations.



$$y(t_0) = \int r(t)\, h(t_0 - t)\, dt$$

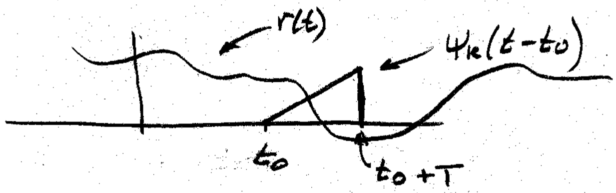
If we make the impulse response a "pre-flipped" version of $\psi_k(t)$, then we get a sliding correlation, or sliding inner product.

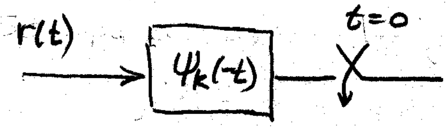




not causal yet, but we'll fix that

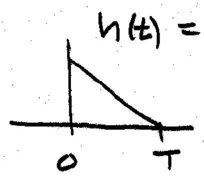
so $y(t_0) = \int r(t)\, h(t_0 - t)\, dt = \int r(t)\, \psi_k(t - t_0)\, dt$



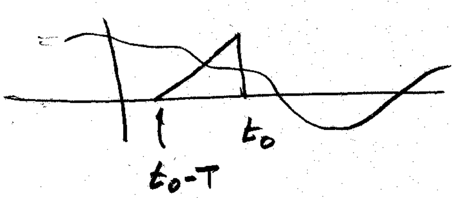


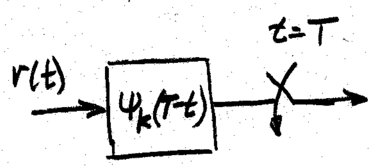
$y(0) = \int r(t)\, \psi_k(t)\, dt$ as desired.

Now make it causal with enough delay (how much is enough?)



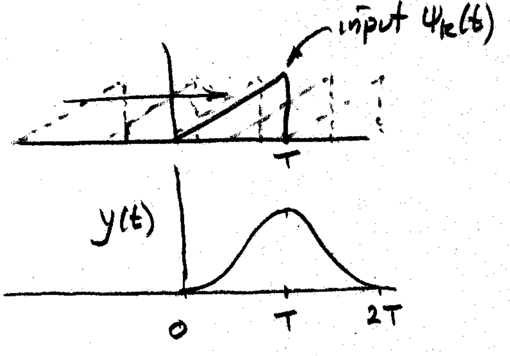
$y(t_0) = \int r(t)\, \psi_k(t - t_0 + T)\, dt$





$y(T) = \int r(t)\, \psi_k(t)\, dt$

The response to $\psi_k(t)$ itself is the pulse autocorrelation (delayed)



input $\Psi_k(f)$

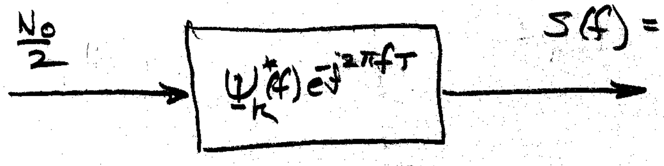
filter $H(f) = \Psi_k(-f)\, e^{-j2\pi f T}$

$= \Psi_k^*(f)\, e^{-j2\pi f T}$

output is

$Y(f) = |\Psi_k(f)|^2 e^{-j2\pi f T}$

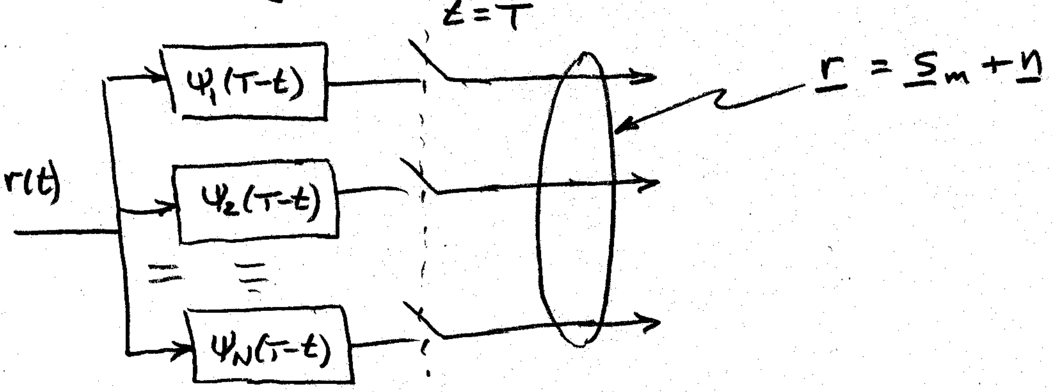
The auto correlation function of the noise at the MF output is proportional to pulse auto correlation function:



$$S(f) = |H(f)|^2 \frac{N_0}{2} = \frac{N_0}{2} |\Psi_k(f)|^2$$

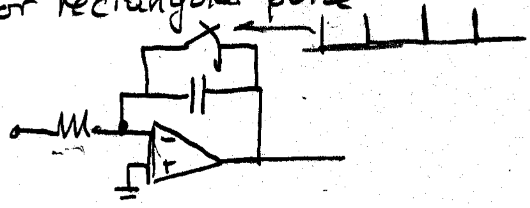
Hence auto correlation function is $R(\tau) = \frac{N_0}{2} R_{\psi_k}(\tau)$

- "Vectorizing" with matched filters

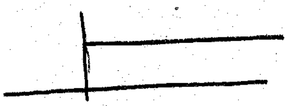


- Analog implementations – "integrate and dump"

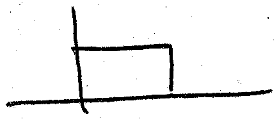
for rectangular pulse



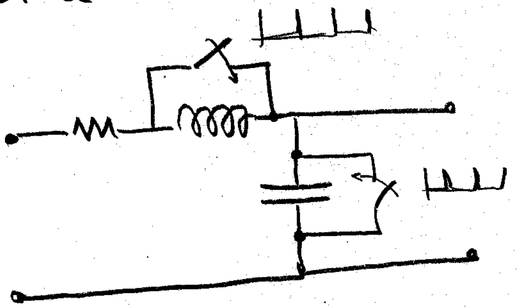
natural $h(t)$



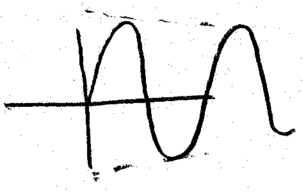
effective $h(t)$



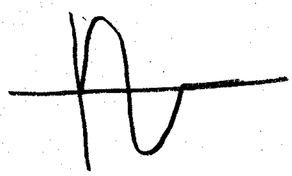
for a tone burst



natural $h(t)$



effective $h(t)$



Making Decisions $\boxed{\text{P+S 7.2.3}}$

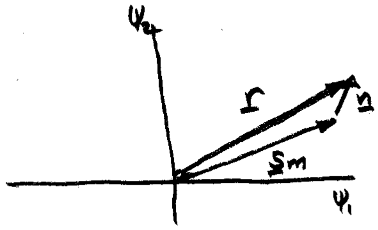
- We now have a length-N vector $\underline{r} = \underline{s}_m + \underline{n}$ as a set of sufficient statistics — but what do we do with it?
- First, we determine the statistics. If we work with $\{\psi_k(t)\}$, the orthonormal basis set, then the noise components satisfy $\overline{n_k n_m} = \frac{N_0}{2}\delta_{km}$

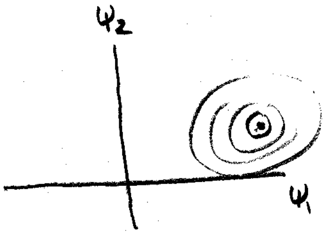
that is, $\overline{\underline{n}\,\underline{n}^T} = \frac{N_0}{2} I = C$ and jointly Gaussian assuming Gaussian white noise

$$p_{\underline{n}}(\underline{n}) = \frac{1}{\sqrt{(2\pi)^N |C|}} \exp\left(-\tfrac{1}{2}\,\underline{n}^T C^{-1} \underline{n}\right) = \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\tfrac{1}{N_0}\,\underline{n}^T \underline{n}\right)$$

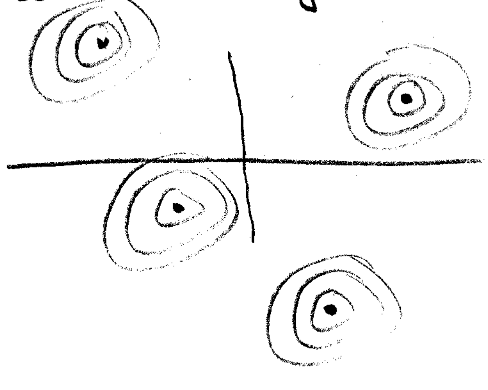
$$= \frac{1}{(\pi N_0)^{N/2}} \exp\left(-\frac{1}{N_0}\sum_{k=1}^{N} n_k^2\right) = \prod_{k=1}^{N} \left(\frac{1}{\sqrt{\pi N_0}} e^{-n_k^2/N_0}\right)$$

Pictures:





with other signals



- Maximum a posteriori probability (MAP) decisions are easily formulated. Given some received $\underline{r}$, pick the signal to

$$\max_i P(\underline{s}_i | \underline{r})$$

or $\max \dfrac{f(\underline{s}_i, \underline{r})}{f(\underline{r})} = \dfrac{f(\underline{r}|\underline{s}_i)P(\underline{s}_i)}{f(\underline{r})}$ — now ignore $f(\underline{r})$ because it doesn't depend on i

or $\max_i f(\underline{r}|\underline{s}_i)P(\underline{s}_i)$ — next, use $\underline{r} = \underline{s}_i + \underline{n}$

or $\max_i f_{\underline{n}}(\underline{r} - \underline{s}_i)P(\underline{s}_i)$

or $\max_i \dfrac{1}{(\pi N_0)^{N/2}} \exp\left(-\dfrac{1}{N_0}|\underline{r} - \underline{s}_i|^2\right) P(\underline{s}_i)$ — now ignore constant, take log

or $\max_i -|\underline{r} - \underline{s}_i|^2 + N_0 \ln(P(\underline{s}_i))$ — expand

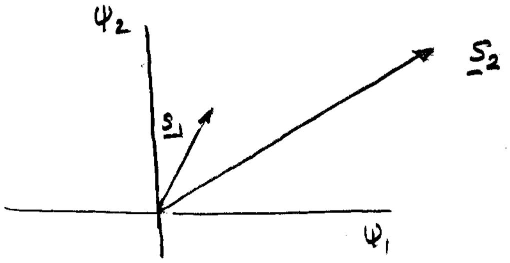
or $\max_i -\left(|\underline{r}|^2 - 2\underline{s}_i^T\underline{r} + |\underline{s}_i|^2\right) + N_0 \ln(P(\underline{s}_i))$ — ignore $|\underline{r}|^2$, note $|\underline{s}_i|^2 = \mathcal{E}_i$

or $$\max_i \underbrace{\underline{s}_i^T \underline{r} + \underbrace{\tfrac{1}{2}\left(N_0 \ln(P(\underline{s}_i)) - \mathcal{E}_i\right)}_{\text{bias } -a_i \quad (a_i \geq 0)}}_{\text{the "metric" } C(\underline{r}, \underline{s}_i)}$$

so the MAP detector calculates $\max_i \underline{s}_i^T \underline{r} - a_i$

In special case of equiprob, equal energy signals, a_i is the same for all, so discard it. Just maximize the correlation $\underline{s}_i^T \underline{r}$.

- Geometric interpretation — start with two signals



Choose $m=2$ if

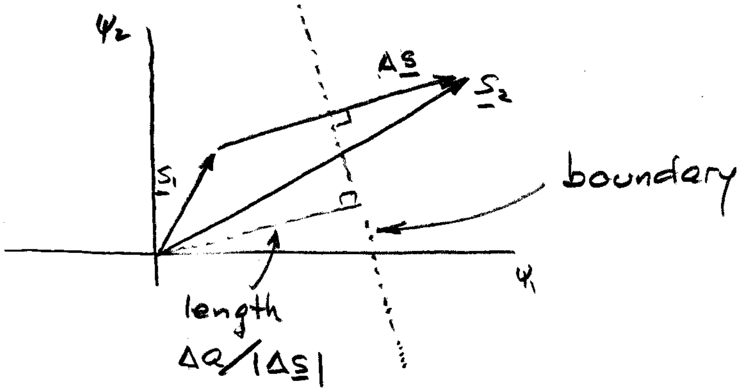
$$C(\underline{r},\underline{s}_2) > C(\underline{r},\underline{s}_1)$$

$$\underline{r}\cdot\underline{s}_2 - a_2 - \underline{r}\cdot\underline{s}_1 + a_1 > 0$$

$$\underline{r}\cdot\Delta\underline{s} - \Delta a > 0 \qquad \Delta\underline{s} = \underline{s}_2 - \underline{s}_1$$

$$\Delta a = a_2 - a_1$$

The boundary $\underline{r}\cdot\Delta\underline{s} + \Delta a = 0$ is a straight line perpendicular to $\Delta\underline{s}$



If the signals are equiprobable, the boundary is the perpendicular bisector. Different probs just shifts it.

In higher dimensions, the boundary is a hyperplane

$$\underbrace{\underline{r}\cdot\frac{\Delta\underline{s}}{|\Delta\underline{s}|}}_{\text{unit normal}} = \underbrace{\frac{\Delta a}{|\Delta\underline{s}|}}_{\text{closest approach.}}$$

With multiple signals, space is partitioned into convex "decision regions"

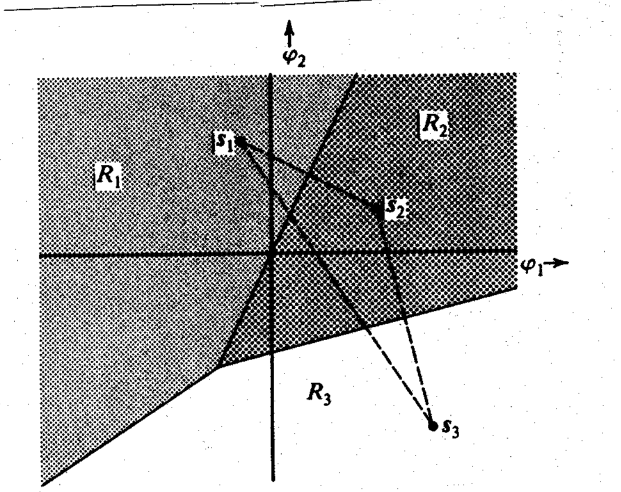


Figure 7.11 Determining optimum decision regions.

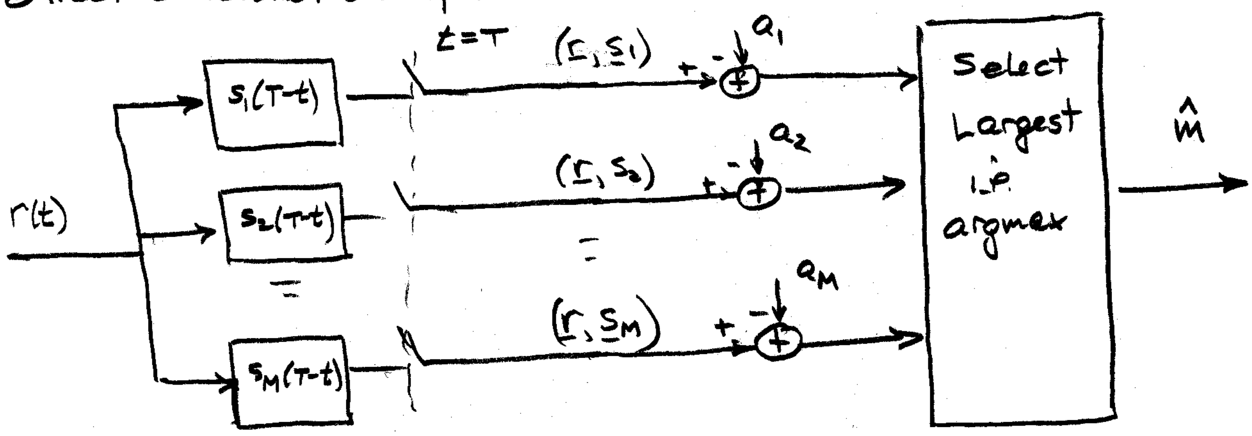
from B.P. Lathi: Modern Digital and Analog Communication Systems

If equiprobable, then just choose the signal the received waveform is closest to.

Implementation Structures

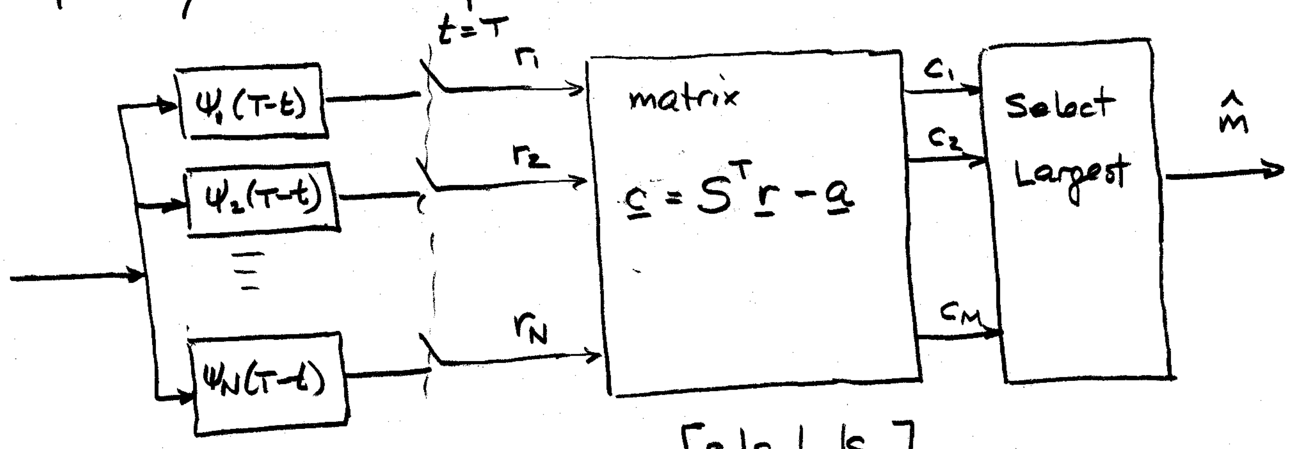
- We have a choice of implementations for the MAP detector, even without consideration of correlator or MF.

- Direct calculation of the metrics (shown with MFs)



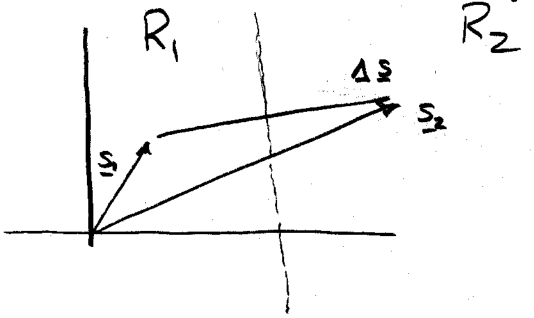
M parallel units, one for each signal

- If M>N, it may be easier to match to the basis functions and do subsequent processing on the "once per symbol" samples



$$S = [\underline{s}_1 | \underline{s}_2 | \cdots | \underline{s}_M]$$

- Simplification for binary $(M=2)$ signals



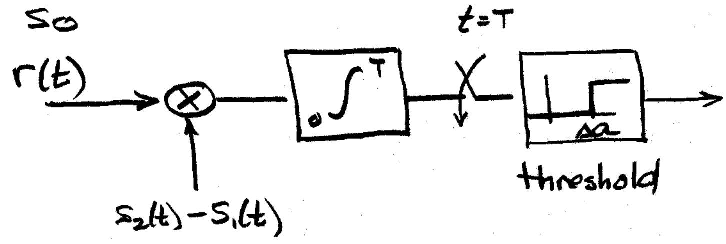
Criterion

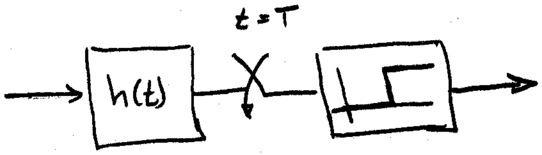
$\underline{r} \cdot \Delta \underline{s} - \Delta a \gtrless 0$

or $\underline{r} \cdot \Delta \underline{s} \gtrless \Delta a$

$\underline{r} \cdot (\underline{s}_2 - \underline{s}_1) \gtrless \Delta a$

So

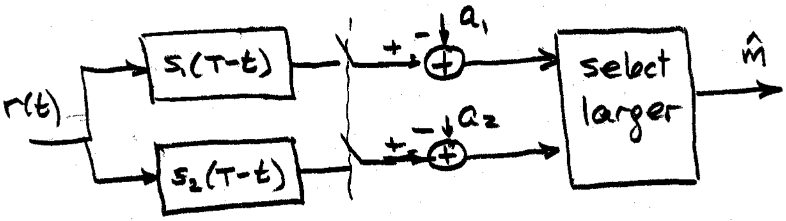




$h(t) = s_2(T-t) - s_1(T-t)$

This is 1-D processing, regardless of apparent number of dimensions!

Another way to see this ... from p. 5.3.9 we have



but "select larger" is the same as "subtract and compare to zero"