5.4 Probability of Error

- In this section, we'll see some common digital signalling methods and derive their error rates in white noise. Bandwidth is also an issue.
- Many apparently dissimilar methods are related simply in signal space — so keep your eyes on the forest, as well as the trees.

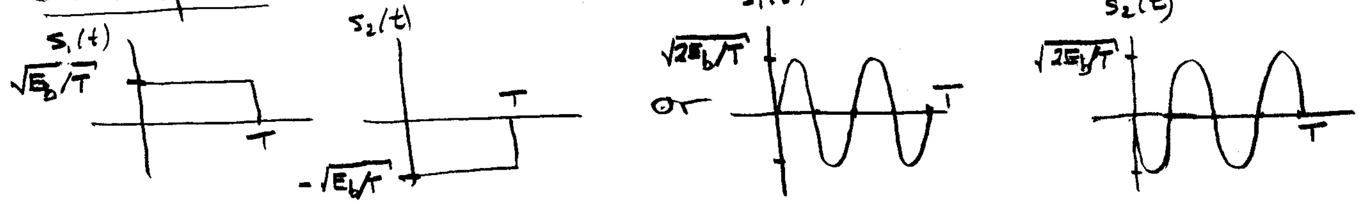
5.4.1 One Dimensional Signals

- Many signal sets are actually one dimensional and can be detected with a single matched filter (p. 5.3.10)

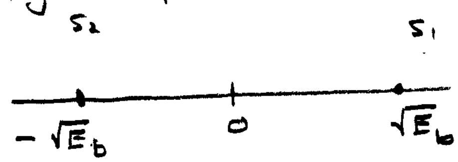
Binary Antipodal

- The classic. Also known as BPSK if the basis function is a sine or cosine.

examples:

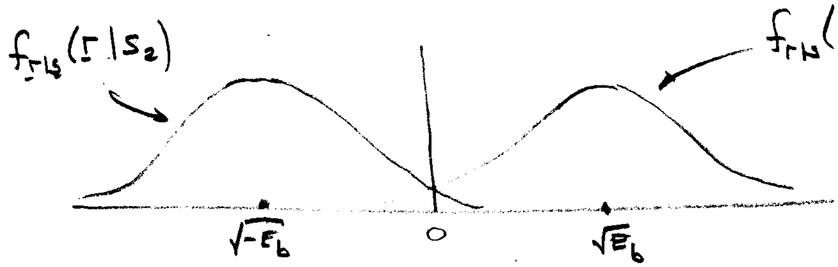


The signal points are



and the conditional pdfs are

$$f_{r|s}(\underline{r}\,|\,\underline{s}_1) = \frac{1}{\sqrt{2\pi}\,\sqrt{N_0/2}}\, e^{-\frac{1}{2}\left(\frac{r-\sqrt{E_b}}{N_0/2}\right)^2}$$



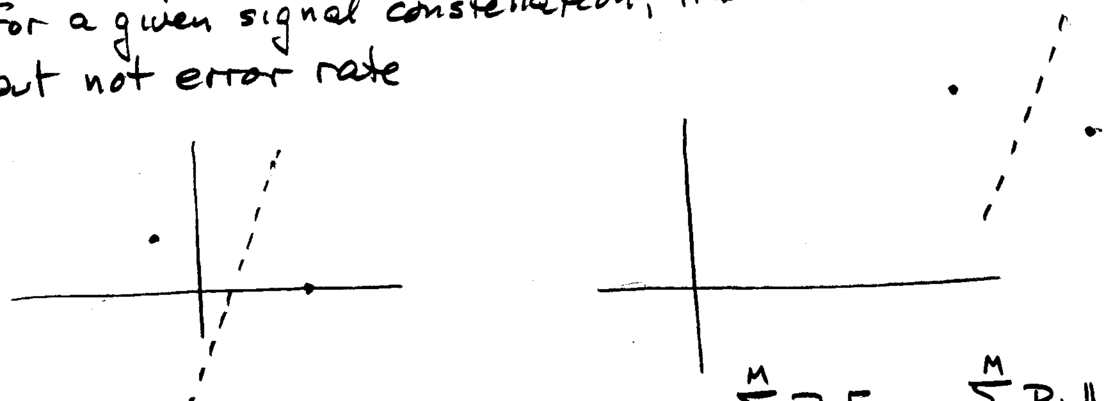
- The probability of error was derived in Section 3.3. For equiprobable signals, it is

$$P_b = Q\left(\frac{m}{\sigma}\right) = Q\left(\frac{\sqrt{E_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{2\gamma_b}\right) \quad \text{binary antipodal}$$

Rotation and Translation

- Simple geometric transformations produce other signal sets.
  - For a given signal constellation, rotation of coordinates does not affect enery or error prob

  - For a given signal constellation, translation affects energy, but not error rate



average energy per symbol $E_s = \sum_{i=1}^{M} P_i E_i = \sum_{i=1}^{M} P_i \|\underline{s}_i\|^2$

after translation by $\underline{\ell}$

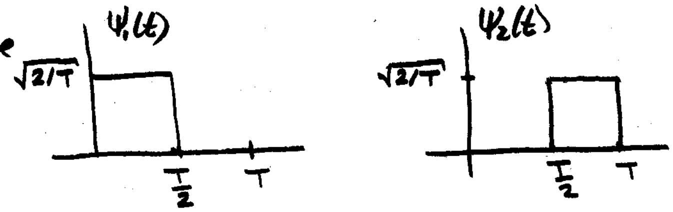
$$E_s' = \sum_{i=1}^{M} P_i \|\underline{s}_i + \underline{\ell}\|^2 = \sum_{i=1}^{M} P_i \left(\|\underline{s}_i\|^2 + 2\underline{\ell}\cdot\underline{s}_i + \|\underline{\ell}\|^2\right)$$

$$= E_s + \|\underline{\ell}\|^2 + 2\underline{\ell}\cdot \underbrace{\sum_{i=1}^{M} P_i \underline{s}_i}_{\text{centroid}}$$

$$= E_s + \|\underline{\ell}\|^2 + 2\underline{\ell}\cdot\underline{s}_c$$

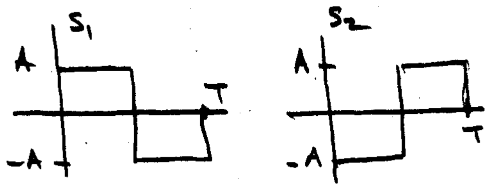
example of rotation, translation.

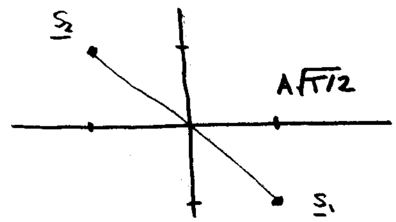
Suppose



basis functions

Manchester "coding"



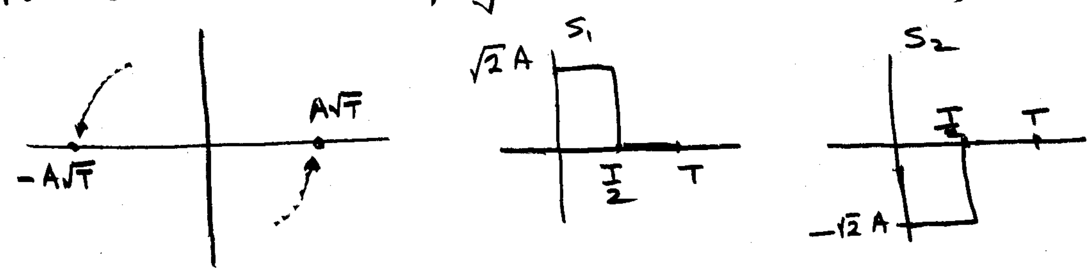


$\|\underline{s}_1\|^2 = A^2 T = E_b$

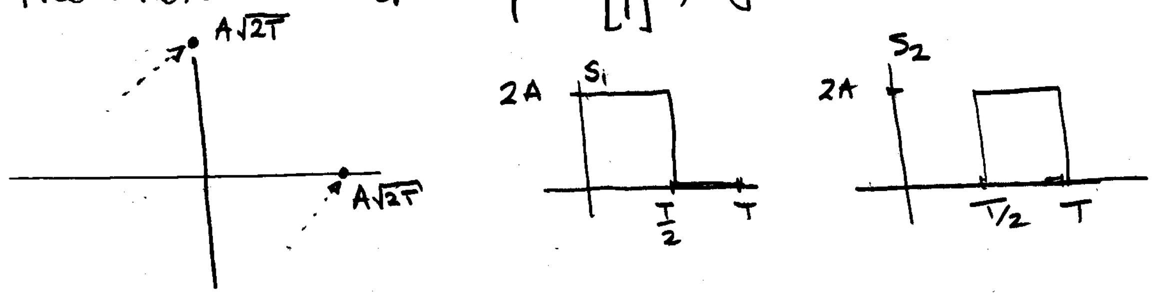
Manchester is useful because no dc component, transitions simplify clock recovery, but more bandwidth.

separation $d = 2A\sqrt{T}$

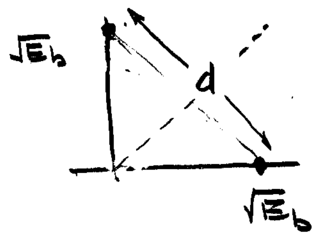
rotate Manchester 45°, get "return to zero" (RZ)



translate Manchester by $A\sqrt{T}\begin{bmatrix}1\\1\end{bmatrix}$, get PPM

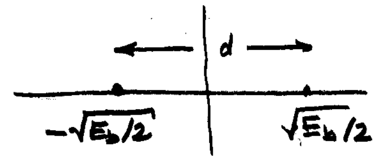


Binary Orthogonal



$d = \sqrt{2E_b}$

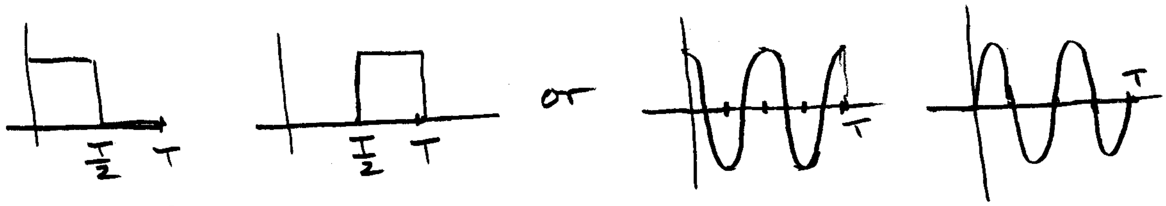
is equivalent to



$$P_b = Q\left(\frac{\sqrt{E_b/2}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\gamma_b}\right)$$

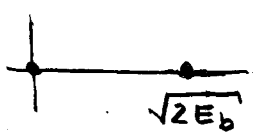
This is 3 dB poorer than binary antipodal, but there are occasional reasons to use it.

examples



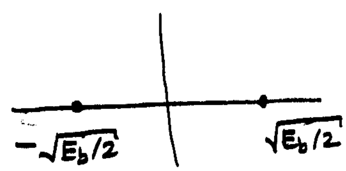
It also occupies more bandwidth than binary antipodal, as we'll see.

On Off Keying (OOK)



$d = \sqrt{2E_b}$

is equivalent to



It's as bad as binary orthogonal $P_b = Q(\sqrt{\gamma_b})$ although it occupies only the bandwidth of binary antipodal. Rarely used now.

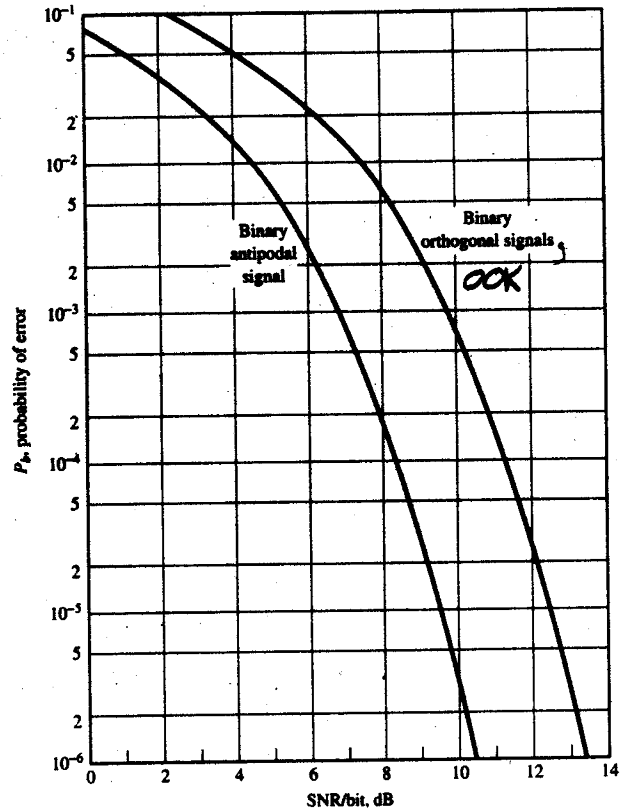
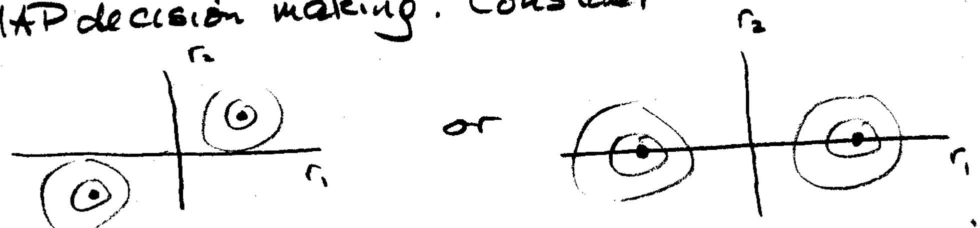


FIGURE 7.42. Probability of error for binary signals.

Note 3dB difference

Irrelevance at Work

- Our "theorem of irrelevance" is a direct consequence of MAP decision making. Consider



(1) Because the direction orthogonal to the signal subspace contains no signal and its noise component is independent of the noise falling in the signal subspace, it is useless for decisions.

(2) By MAP approach, the decision boundary is parallel to the orthogonal complement, so received components in that direction have no effect.

(3) Formally, for •—|—•

$$\max_i \; -|\underline{r}-\underline{s}_i|^2 + N_o \ln(P(\underline{s}_i))$$

$$\max_i \; -(r_1-\underline{s}_i)^2 - r_2^2 + N_o \ln(P(s_i))$$

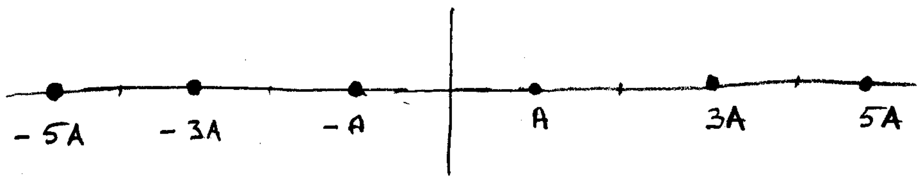
r_2 common to both candidates

$$\min_i \; (r_1-\underline{s}_i)^2 - N_o \ln(P(\underline{s}_i))$$

and reverse sign and max/min

Multilevel Signals

- Still one dimensional, but M-ASK or M-PAM



M=6 here

- Be very careful:

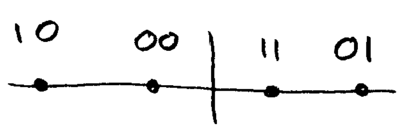
  E_s or E_b? γ_s or γ_b?

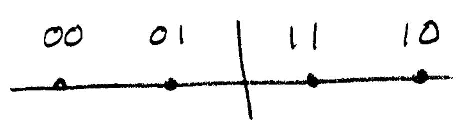
  P_b or P_s?

$$E_s = \frac{2}{M}\sum_{i=1}^{M/2}(2i-1)^2A^2 = \frac{A^2}{3}(M^2-1)$$

$$E_b = E_s / \log_2(M)$$

- The relation between BER and SER depends on the assignment of bits to constellation points.





Grey coding — only one bit changes in a "nearest-neighbour" error.

e.g.

10	100
11	101
01	111
00	110
	010
	011
	001
	000

to contrive

$$G(n) = \begin{bmatrix} 1 & G(n-1) \text{ reversed} \\ \hline 0 & G(n-1) \end{bmatrix}$$

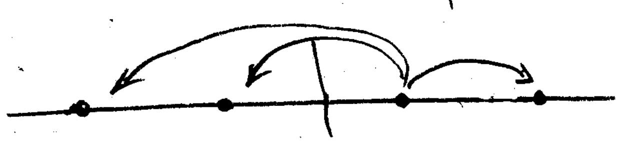
- Prob of symbol error

$$P_s = \begin{cases} 2\,Q\left(\frac{A}{\sqrt{N_0/2}}\right) & \text{inside points, prob } \frac{M-2}{M} \\ Q\left(\frac{A}{\sqrt{N_0/2}}\right) & \text{outside points, prob } \frac{2}{M} \end{cases}$$

$$= \frac{2(M-1)}{M} Q\left(\frac{A}{\sqrt{N_0/2}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{M^2-1}\cdot\frac{E_s}{N_0}}\right) = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{M^2-1}\cdot\gamma_s}\right)$$

- Prob of bit error?

Messy — noise causes constellation point to be decided as any of the others, with various probabilities.



But at moderate to high SNR (i.e. reasonably low error rate) the nearest neighbour errors will dominate.

With Grey coding, the BER ≈ SER

$$P_b \approx \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6}{M^2-1}\gamma_s}\right)$$

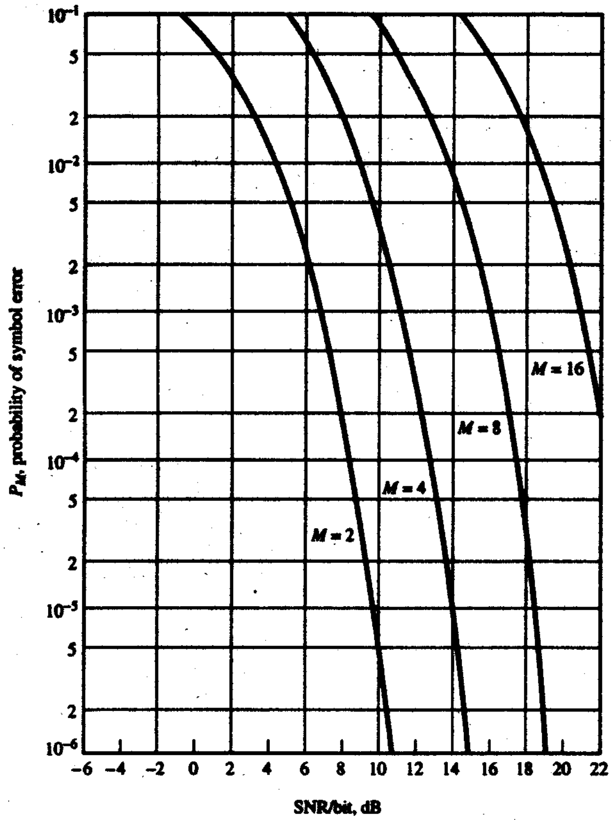


FIGURE 7.44. Probability of a symbol error for PAM.

How many ways can you misunderstand this graph?

Every doubling of M (every additional bit) causes the same incremental SNR penalty:

$$P_s = \frac{2(M-1)}{M}\, Q\left(\sqrt{\frac{6}{M^2-1}\gamma_s}\right) \sim 2\,Q\left(\sqrt{6\,\frac{\gamma_s}{M^2}}\right)$$

To keep a given P_s as M changes,

$$\gamma_s \propto M^2 \qquad \gamma_{s\,dB} = c + 20\log_{10} M$$

Doubling M costs 6 dB, asymptotically