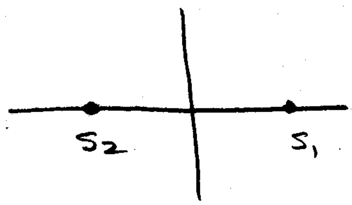
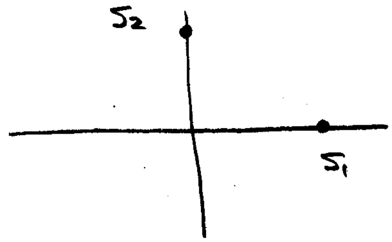
5.4.2 Multidimensional Signals

Bandwidth

- When we transmit genuinely multidimensional signals, rather than a constellation on a 1-D subspace that includes the origin, we have to acknowledge the increased bandwidth.





Bandwidth is determined by energy spectral density since these are one shot pulses.

$$S_s(f) = \sum_{i=1}^{M} |S_i(f)|^2 P(s_i)$$

For 1-D, $S_s(f) = |S_1(f)|^2 \sum_i \frac{E_i}{E_1} P(s_i)$ — all have same shape ppl to $s_1(t)$

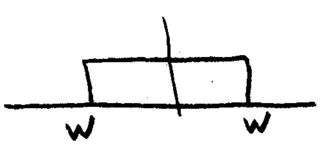
For 2-D, $S_s(f) = \frac{1}{2} \sum_{i=1}^{2} |S_i(f)|^2$ — equiprob

where $\int S_1(f) S_2^*(f)\, df = 0$

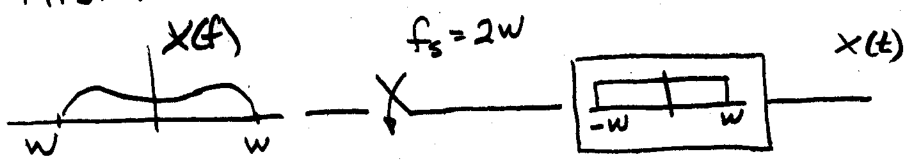
Does the increased dimensionality increase the required bandwidth?

Rule of thumb: In a bandwidth W and time T, we can pack about

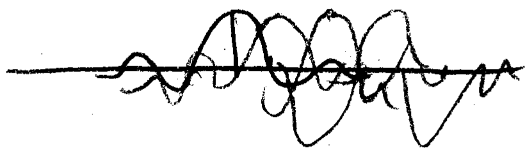
$N = 2WT$ orthogonal functions



First illustration:

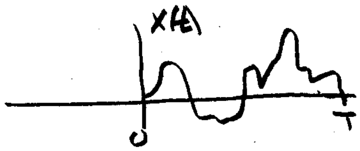


$$x(t) = \sum_k x(k/2W)\,\mathrm{sinc}\big(2W(t - k/2W)\big)$$



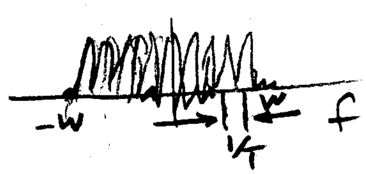
- The sinc functions are orthogonal.
- There are $f_s T = 2WT$ of them in some time T, neglecting end effects
- Since they are sufficient to represent $x(t)$, faster sampling does not generate more.

Second illustration:



$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}$$

In a bandwidth W, we get approximately $2W/(1/T) = 2WT$ components X_i. Each has Re and Im parts, which suggests $4WT$ independently selectable numbers (dimensions). However, conjugate



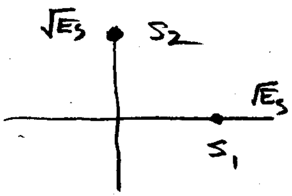
symmetry makes $X_{-i} = X_i^*$, so it's back to $2WT$ dimensions.

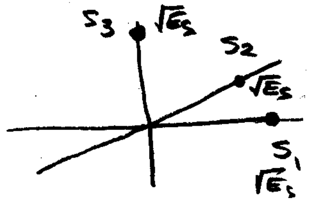
Conclusion: If dimensionality of our size-M constellation is N, then the product WT must be at least N/2

$$WT \geq N/2$$

Larger N $\Rightarrow$ send more slowly, or use more bandwidth, or both

M-Orthogonal Signals

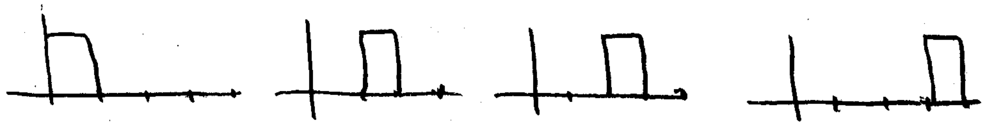




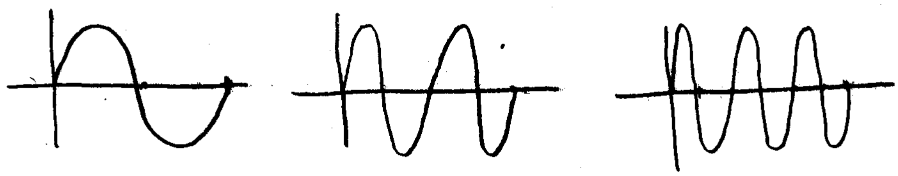
etc

$M = N$

e.g. PPM

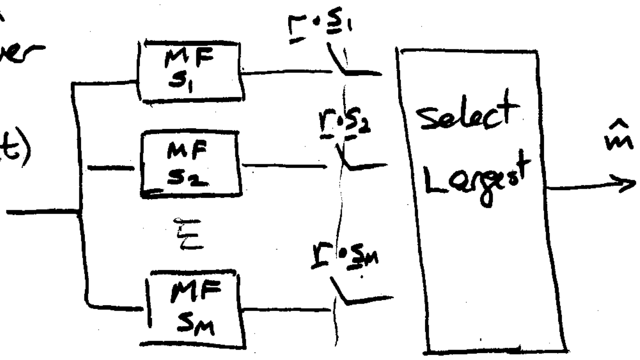


FSK



- All signals are equidistant, all are nearest neighbours, any error is as likely as another.
  Distance $d = \sqrt{2E_s}$

- Receiver



$$\begin{array}{cccc}
\underline{s}_1 & \underline{s}_2 & \underline{s}_3 & \underline{s}_M \\
\sqrt{E_s} & 0 & 0 & 0 \\
0 & \sqrt{E_s} & 0 & 0 \\
0 & 0 & \sqrt{E_s} & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \sqrt{E_s}
\end{array}$$

If $\underline{s}_1$ sent, then $\underline{r} = [\sqrt{E_s}+n_1, n_2, \cdots n_M]^T$

Form $\underline{r}\cdot\underline{s}_1 = \sqrt{E_s}\,(\sqrt{E_s}+n_1)$

$\underline{r}\cdot\underline{s}_2 = \sqrt{E_s}\,(n_2)$

$\vdots$

$\underline{r}\cdot\underline{s}_M = \sqrt{E_s}\,(n_M)$

common factor – ignore it

- Given r_1, the prob of no symbol error is

$$P(c \mid r_1) = \Pr\left[n_2 < r_1 \text{ and } n_3 < r_1 \text{ and } \cdots \text{ and } n_M < r_1\right]$$
$$= \left(1 - Q\left(\frac{r_1}{\sqrt{N_0/2}}\right)\right)^{M-1}.$$

and unconditional prob

$$P(c) = \int P(c \mid r_1)\, f_{r_1}(r_1)\, dr_1 = \int P(c \mid r_1)\, f_n(r_1 - \sqrt{E_s})\, dr_1$$
$$= \int_{-\infty}^{\infty} \left(1 - Q\left(\sqrt{\frac{2 r_1^2}{N_0}}\right)\right)^{M-1} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{N_0/2}} e^{-\frac{1}{2}\cdot\frac{2}{N_0}(r_1 - \sqrt{E_s})^2}\, dr_1$$

and symbol error rate is

$$P_s = 1 - P(c)$$

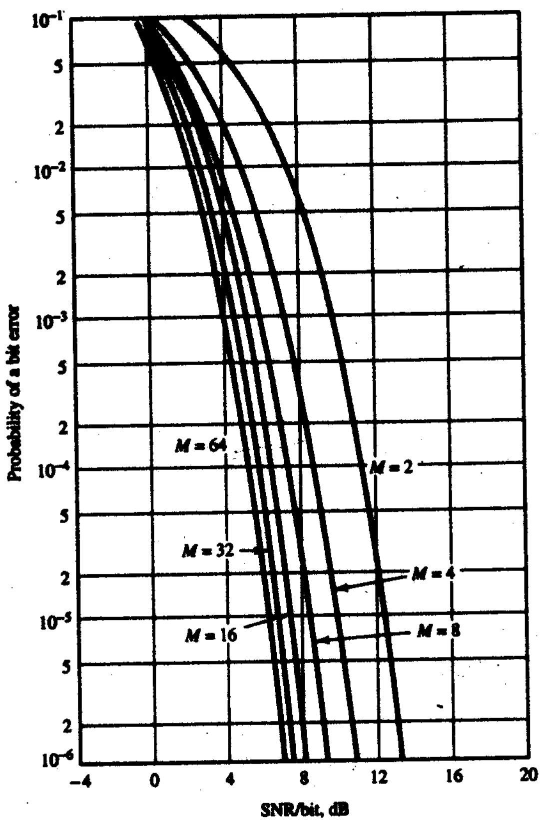


FIGURE 7.45. Probability of bit error for coherent detection of orthogonal signals.

- Bit error rate:
  - No point in fancy mappings.
  - Any particular error has prob $P_s/(M-1) = \dfrac{P_s}{2^k - 1}$ (k bits)
  - n bit errors can be had in $\binom{k}{n}$ ways, so

$$P_b = \frac{P_s}{2^k - 1} \cdot \sum_{n=1}^{k} n \binom{k}{n}$$
$$= k\, \frac{2^{k-1}}{2^k - 1} \cdot P_s \approx \frac{k}{2} P_s$$

It improves as M increases (more on this in the coding section) but the WT product increases, too.

- An approximate SER for othogonal signals is available through the union bound. Suppose $\underline{s}_1$ is transmitted, and E_i is the event that $\underline{r}$ is closer to $\underline{s}_i$ than to $\underline{s}_1$. It results in an error, and its probability is just that of binary orthogonal.

$$P_2 = P(E_i) = Q(\sqrt{\gamma_s})$$

There are M-1 such events, and the overall SER

$$P_s = \text{Prob}[E_2 \cup E_3 \cup \cdots \cup E_M]$$

They are not mutually exclusive events — $\underline{r}$ could be closer to both $\underline{s}_2$ and $\underline{s}_3$ than to the correct $\underline{s}_1$.

Consequently, the sum of the pairwise event probabilities is an upper bound.

$$P_s \leq \sum_{i=2}^{M} P(E_i) = (M-1)\,Q(\sqrt{\gamma_s})$$

It gets tighter as γ_s becomes large.