8.5 Error Performance of Block Codes $\boxed{\text{P+S } 10.5.1}$

- We already have, for t-error correction

$$P_e' = \sum_{k=t+1}^{n} \binom{n}{k} P_b^k (1-P_b)^{n-k} \qquad \text{WER in terms of BER}$$

Resorting to a bound, instead of this exact expression, lets us compare hard decision decoding (what we have been doing so far) and soft decision decoding (TBA).

- Use a union bound. Clearly, the pairwise error prob between a pair of codewords depends on the Hamming distance, since bit positions in which the two words agree carry no information. If $\underline{c}_i$ is sent, then

$$P_e'(\underline{c}_i) \leq \sum_{j \neq i} \underbrace{P_2(\underline{c}_i, \underline{c}_j)}_{\substack{\text{prob } \underline{c}_j \text{ is} \\ \text{preferable to} \\ \underline{c}_i}} = \sum_{d=d_{min}}^{n} \underbrace{\nu_d(\underline{c}_i)}_{\substack{\#\text{codewords} \\ \text{distance } d \text{ from} \\ \underline{c}_i}} \underbrace{P_2(\underline{c}_i, d)}_{\substack{\text{prob that a codeword} \\ d \text{ distant from } \underline{c}_i \\ \text{is preferable}}}$$

But in linear codes, $\underline{c}_k + \underline{c}_m \in \mathcal{C}$ for any k, m, so $\nu_d(\underline{c}_i)$ and $P_2(\underline{c}_i, d)$ are the same for all $\underline{c}_i$. Therefore

$$P_e' \leq \sum_{d=d_{min}}^{n} \nu_d P_2(d) \leq (2^k - 1) P_2(d_{min})$$

- Hard decision decoding: make hard (0,1) decisions on the coded bits, then rely on code structure to correct (reverse) some of them.

  - Get the pairwise error prob. In comparing two codewords $\underline{c}_i$, $\underline{c}_j$ to the received $\underline{r}$, we need look only at the d positions in which they differ.

true $\underline{c}_i$	x x	0	x x x x	1	x	1	x x
false $\underline{c}_j$	x x	1	x x x x	0	x	0	x x

x's are same in the two words.

  - with BER P_b, the prob that $\underline{r}$ is closer to $\underline{c}_j$ than $\underline{c}_i$ is

$$P_{2h}(d) = \begin{cases} \displaystyle\sum_{i=\frac{d+1}{2}}^{d} \binom{d}{i} P_b^i (1-P_b)^{d-i}, & d \text{ odd} \\ \displaystyle\sum_{i=\frac{d+2}{2}}^{d} \binom{d}{i} P_b^i (1-P_b)^{d-i} + \frac{1}{2}\binom{d}{d/2} P_b^{d/2} (1-P_b)^{d/2} \end{cases}$$

h: hard

count ones at the midpoint as 50/50

$$\leq \sum_{i=\lfloor\frac{d+1}{2}\rfloor}^{d} \binom{d}{i} P_b^i (1-P_b)^{d-i}$$

  - We can expand and approximate

$$P_{2h}(d) \leq \sum_{i=\lfloor(d+1)/2\rfloor}^{d} \binom{d}{i} \left(\frac{P_b}{1-P_b}\right)^i (1-P_b)^d \qquad \frac{P_b}{1-P_b} \approx P_b$$

hard

$$\approx \sum_{i=\lfloor(d+1)/2\rfloor}^{d} \sum_{m=0}^{d} \binom{d}{i}\binom{d}{m}(-1)^m P_b^{i+m}$$

- Next, assume binary antipodal, coherent detection if bandpass,

$$P_b = Q\left(\sqrt{2\frac{k}{n}\gamma_b}\right) = Q\left(\sqrt{2\gamma_c}\right)$$

so

$$P_{2h} \approx \sum_{i=\lfloor\frac{d+1}{2}\rfloor}^{d} \sum_{m=0}^{d} \binom{d}{i}\binom{d}{m}(-1)^m \, Q\left(\sqrt{2\gamma_c}\right)^{i+m}$$

- Finally, overbound $Q(x) \leq \frac{1}{2} e^{-x^2/2}$, so

$$P_{2h}(d) \lesssim \sum_{i=\lfloor\frac{d+1}{2}\rfloor}^{d} \sum_{m=0}^{d} \binom{d}{i}\binom{d}{m}(-1)^m\left(\frac{1}{2}\right)^{i+m} e^{-\gamma_c(i+m)}$$

dominated by the $i=\lfloor\frac{d+1}{2}\rfloor$, $m=0$ term as $\gamma_c \gg 1$

so

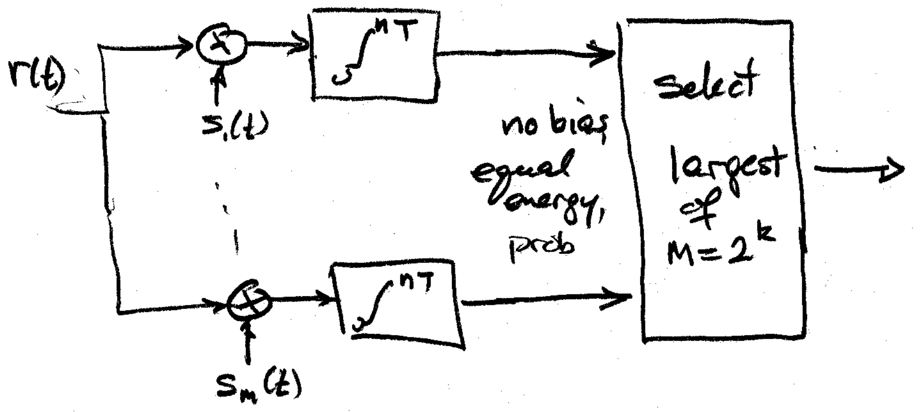
$$P_{2h}(d) \lesssim \binom{d}{\lfloor\frac{d+1}{2}\rfloor}\left(\frac{1}{2}\right)^{\lfloor\frac{d+1}{2}\rfloor} e^{-\gamma_c \lfloor\frac{d+1}{2}\rfloor}$$

e.g. if $d=5$, then

$$P_{2h}(5) \lesssim \sum_{i=3}^{5} \sum_{m=0}^{5} \binom{5}{i}\binom{5}{m}(-1)^m\left(\frac{1}{2}\right)^{i+m} e^{-\gamma_c(i+m)}$$

$$\approx \binom{5}{3} 2^{-3} e^{-3\gamma_c} = \frac{5}{2} e^{-3\gamma_c}$$

- Now soft decisions. Imagine the code words as multidimensional points $\underline{s}_i$, $i = 1..M$ in Euclidean space, and detect with correlators, as in Section 5.



- The pairwise prob depends on Hamming distance, as in hard decisions, and

$$P_e' \leq \sum_{d=d_{min}}^{n} \nu_d P_{2s}(d) \leq (2^k - 1) P_{2s}(d_{min})$$

- The pairwise prob is determined by the Euclidean distance in the difference $s_i(t) - s_j(t)$. After vectorizing

$\underline{s}_i$ × × × $\sqrt{E_c}$ × × $-\sqrt{E_c}$ × $-\sqrt{E_c}$ × ×

$\underline{s}_j$ × × × $-\sqrt{E_c}$ × × $\sqrt{E_c}$ × $\sqrt{E_c}$ × ×

binary antipodal

Squared Euclidean distance

$$(d^E)^2 = (\pm 2\sqrt{E_c})^2 d = 4E_c d$$

- Hence

$$P_{2s}(d) = Q\left(\frac{d^E/2}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{2\gamma_c d}\right) \leq \frac{1}{2} e^{-\gamma_c d}$$

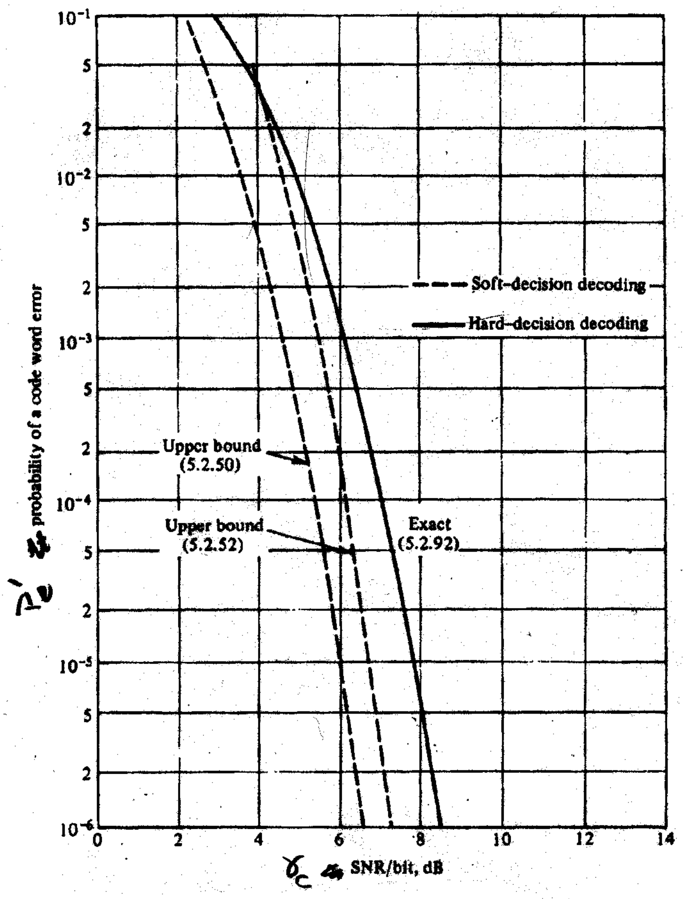
- Comparison

$$P'_{es} \leq \sum_{d=d_{min}}^{n} \nu_d P_{2s}(d) \leq (2^k - 1) P_{2s}(d_{min}) \lesssim 2^{k-1} e^{-\gamma_c d_{min}}$$

$$P'_{eh} \leq \sum_{d=d_{min}}^{n} \nu_d P_{2h}(d) \leq (2^k - 1) P_{2h}(d_{min}) \sim \binom{d_{min}}{\left\lfloor \frac{d_{min}+1}{2} \right\rfloor} 2^{-\left\lfloor \frac{d_{min}+1}{2} \right\rfloor} e^{-\gamma_c \left\lfloor \frac{d_{min}+1}{2} \right\rfloor}$$

- The exponent is twice as large (or a little less) in soft decoding! When this and constants are accounted for, typically about 2 dB better with soft decision.
- The cost: complexity. There are 2^k correlators. OK for (7,4). Not OK for (63,45). Rarely used for block codes.
- But... soft decoding turns out to be easy in Viterbi decoding of convolutional codes.

– Comparison of soft and hard decisions with the (23, 12) Golay code (a perfect code) with $t = 3$, $d_{min} = 7$



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hard (5.2.92) $\quad P'_{eh} = \sum_{k=4}^{23} \binom{23}{k} P_e^{k} (1-P_e)^{23-k}$

soft (5.2.52) $\quad P'_{es} = 2^{12} P_{2S}(7) = 2^{12} Q\left(\sqrt{14\gamma_c}\right)$

soft (5.2.50) $\quad P'_{es} = \sum_{d=7}^{23} \nu_d P_{2S}(d) = \sum_{d=} \nu_d Q\left(\sqrt{2d\gamma_c}\right)$