using (64) and (66),

$$E\langle x(t)\rangle = \frac{1}{T}\sum_{n=-\infty}^{\infty} \exp\left(2\pi i \frac{n}{T} t - \frac{1}{2} R_0 \left[\frac{2\pi n}{T}\right]^2\right) F\left(\frac{n}{T}\right), \tag{67}$$

which is a periodic function with period T. It can be shown that $x(t) - E\langle x(t)\rangle$ is expressible in the form (31). Omitting the details of calculation we give the corresponding matrix-valued spectral density,

$$F_{n,n'}(\lambda) = G\left(\frac{n}{T} + \lambda, \frac{n'}{T} + \lambda\right) F\left(\frac{n}{T} + \lambda\right) F\left(\frac{n'}{T} + \lambda\right) \tag{68}$$

where

$$G(\lambda,\lambda') = \frac{1}{T}\sum_{m=-\infty}^{\infty} \exp\left[2\pi i m T\lambda\right]\left[\exp\left(4\pi^2 R_m \lambda\lambda'\right) - 1\right] \times \exp\left[-2\pi^2 R_0(\lambda^2 + \lambda'^2)\right]. \tag{69}$$

In particular if I_m is an uncorrelated random sequence, i.e., $E\langle I_m I_n\rangle = \delta_{mn} R_0$, (69) takes a simpler form.

ACKNOWLEDGMENT

The author would like to express his gratitude to Prof. J. Ikenoue for constant encouragement and also to the colleagues of his laboratory for valuable discussions.

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Complex Gaussian Noise Moments

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Abstract—The problem of the computation of moments of nonzero mean circularly complex Gaussian noise is treated. The noise need not be symmetric about the carrier frequency. In particular, the second-order moments are computed, and expansions are given. The necessary univariate and bivariate complex Hermite polynomials are discussed. The means of some rational functions useful in FM theory are given. This paper extends work of Rice, Middleton, and Zadeh to complex Gaussian noise with nonzero mean and nonsymmetrical power spectrum.

I. INTRODUCTION

THE computation of means of Gaussian random processes is very common and has an extensive history. The essentials of these computations were given by Rice [1], and these were further developed by Wang [2], and presented in perhaps their greatest detail by Middleton [3]. For carrier signals the technique was essentially to treat the problem by an extension of the techniques used for real signals. Recently it has become clear that the economy of thought and writing accompanying the casting of computations into ones involving complex signals makes carrier system analysis much easier—it is analogous to doing network analysis using complex numbers as opposed to using sines and cosines. In the process of reinterpreting the usual results using the complex representation, it has become clear that it is relatively easy to extend many results, and furthermore that it is easy to include signals that are not placed symmetrically in the noise.

In this paper, we have summarized the main formulas associated with moments of circularly complex Gaussian noise processes with nonzero means. However, this moment theory omits several parts of the theory of such processes: the zero-crossing problem, the problem of the distribution of maximums, and error-rate computations. These involve means of functions that are not analytic.

We first review the Mehler expansions for a pair of correlated variables and the associated complex Hermite polynomials. The generalization to Hermite functions is made to allow means of rational functions. This work was presented earlier in [4]. Then certain expansions of functions that converge in the Gaussian norm are given. The complex Grad polynomials are derived.

The problem of the computation of the means of functions involving bivariate complex Gaussian random variables is then given. These lead, for moments, to generalized Hermite polynomials, similar to those treated by Appell and Kampé de Feriet [14]. The Reed [5] and Middleton [3] expansions for these polynomials are presented. Some

Manuscript received March 12, 1970; revised September 8, 1970.
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special rational functions useful in FM theory are derived.

In this paper, we have attempted to present the results. There are indications of the derivations of the more important formulas. The techniques used are either symbolic calculus or integration using Watson [6]. No use is made of contour integration. The rest of the Introduction summarizes the elementary properties of complex Gaussian noise.

The carrier signal $s(t)$ may be written as the real part of a complex function

$$s(t) = \mathrm{Re}\,[z(t)\exp(j2\pi f_c t)].$$

In this representation, we have used the complex carrier signal $z(t)$ and the carrier frequency f_c. Noise that is bandpass in nature may be represented by using the circularly complex Gaussian noises z_i. These noises are characterized by their means $\bar{z}_i$, their covariances

$$\Lambda_{ij} = E[(z_i - \bar{z}_i)(z_j - \bar{z}_j)^*/2], \tag{1}$$

the property of being circularly complex, i.e., that

$$E[(z_i - \bar{z}_i)(z_j - \bar{z}_j)/2] = 0 \tag{2}$$

(henceforth we drop the adverb circularly), and with a characteristic function

$$\psi = \exp(-\rho^\dagger\Lambda\rho/2 + j\,\mathrm{Re}\,\rho^\dagger\bar{z}) \tag{3}$$

where, as usual, * indicates (usually) complex conjugation, † is the Hermitian conjugation of a vector or matrix. Λ is the positive definite matrix with elements the covariances, z and ρ are vectors with elements $z_i = x_i + jy_i$ and $\rho_i = Ui + jv_i$, and $\bar{z}_i$ is the expectation of the random variable z_i. The density function itself is

$$p(x_1,\cdots,y_n) = \exp[-(z-\bar{z})^\dagger\Lambda^{-1}(z-\bar{z})/2]/(2\pi)^n \det\Lambda. \tag{4}$$

Miller, in a review paper [17], has presented a discussion of the representation theorems, of generating functions, certain moments, and 32 references about complex Gaussian noise.

It will be useful to define certain complex derivatives in terms of real derivatives.

$$\begin{aligned}\partial/\partial z &= (\partial/\partial x - j\partial/\partial y)/2\\ \partial/\partial z^* &= (\partial/\partial x + j\partial/\partial y)/2.\end{aligned} \tag{5}$$

These derivatives satisfy $\partial z/\partial z^* = \partial z^*/\partial z = 0$ and also the relationship

$$\partial(1/z)/\partial z^* = \partial(1/z^*)/\partial z = \pi\,\delta(x)\,\delta(y). \tag{6}$$

We occasionally use the following integration by parts formula

$$\begin{aligned}&\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy g(z,z^*)(\partial/\partial z)^n(\partial/\partial z^*)^m h(z,z^*)\\ &= (-1)^{n+m}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy h(z,z^*)(\partial/\partial z)^n(\partial/\partial z^*)^m g(z,z^*)\end{aligned} \tag{7}$$

satisfied by functions that vanish appropriately at infinity.

To be consistent with the usual definition of characteristic function, we have used the following Fourier series mates.

$$g(\rho) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dy_n f(z)\exp(j\,\mathrm{Re}\,\rho^\dagger z) \tag{8a}$$

and

$$f(z) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} du_1\cdots dv_n g(\rho)\exp(-j\,\mathrm{Re}\,\rho^\dagger z)/(2\pi)^{2n}. \tag{8b}$$

The Parseval formula is

$$\begin{aligned}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} du_1\cdots dv_n g_1(\rho)g_2^*(\rho)/(2\pi)^{2n}\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dy_n f_1(z)f_2^*(z).\end{aligned} \tag{9}$$

The Fourier transform of $1/z$ is $2\pi j/\rho$. The symbolic proof is interesting; we have

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy \exp[j(xu + yv)]/z\\ &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy/z(\partial/\partial z^*)\{\exp(j\,\mathrm{Re}\,\rho^* z)/j\rho\}\\ &= 2j/\rho\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\,dy\exp(j\,\mathrm{Re}\,\rho^* z)\,\partial/\partial z^*(1/z) = 2\pi j/\rho\end{aligned}$$

using the earlier results for derivatives and integrating by parts.

Similarly, the transform of z itself is the operator

$$2j(2\pi)^2\,\delta(u)\,\delta(v)\,\partial/\partial\rho^*. \tag{10}$$

Application of the Parseval formula to the mean of the function f with Fourier transform g yields Rice's method for the computation of means of Gaussian random processes

$$\begin{aligned}&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dy_n f(z)\,p(x_1,\cdots,y_n)\\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} du_1\cdots dv_n g(\rho)\exp[-j\,\mathrm{Re}\,(\rho^\dagger\bar{z})\\ &\qquad - \rho^\dagger\Lambda\rho/2]/(2\pi)^{2n}.\end{aligned} \tag{11}$$

The advantage of this method is that the computation in this form is often easier.

In what follows, we will make occasional use of the Fourier–Gauss transform of a function

$$\begin{aligned}f(z) &= \exp(z^\dagger\Lambda^{-1}z/2)\det\Lambda(2\pi)^n\int_{-\infty}^{\infty} du_1\cdots dv_n\\ &\quad\times g(\rho)\exp[-j(\rho^\dagger z + z^\dagger\rho)/2 - \rho^\dagger\Lambda\rho/2]/(2\pi)^{2n}\end{aligned} \tag{12}$$

and, in this connection, we will use the following result. If the function f is only a function of the variables z_1 and z_1^*, then the bivariate Fourier–Gauss transform f_2 may be expressed in terms of the univariate Fourier–Gauss transform f_1 by the following rule:

$$f_2(z_1,z_2,\Lambda_{11},\Lambda_{12},\Lambda_{21},\Lambda_{22}) = f_1(\Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}, z_1 - \Lambda_{12}\Lambda_{22}^{-1}z_2, z_1^\dagger - z_2^\dagger\Lambda_{22}^{-1}\Lambda_{21}) \quad (13)$$

where

$$f_1(\Lambda,z,z^\dagger) = \exp(z^*\Lambda^{-1}z/2)\det\Lambda(2\pi)\int_{-\infty}^{\infty} du\,dv/(2\pi)^2 \times g(\rho_1)\exp[-j(\rho^\dagger z + z^\dagger\rho)/2 - \rho^\dagger\Lambda\rho/2].$$

Lack of correlation implies independence. Suppose a set of Gaussian random variables is partitioned into the two vectors $Z_a = (z_1,z_2,\cdots,z_n)$ and $Z_b = (z_{n+1},\cdots,z_N)$, with similar partitioning of the correlation matrix Λ into Λ_{aa}, Λ_{ab}, Λ_{ba}, and Λ_{bb}. Then the best mean-square estimate of the vector Z_a conditioned on the vector Z_b is $\bar{Z}_a + \Lambda_{ab}\Lambda_{bb}^{-1}(Z_b - \bar{Z}_b)$ and Z_a less this latter vector is uncorrelated with any of the elements of the vector Z_b. The conditioned (on Z_b) covariance matrix of the elements of the vector $Z_a - \bar{Z}_a - \Lambda_{ab}\Lambda_{bb}^{-1}(Z_b - \bar{Z}_b)$ is $\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba}$.

There are two general results on the dependence of means of functions of complex Gaussian random variables often used in practice—the Bonnet theorem and the Price theorem. The Bonnet theorem relates the derivative of a mean with respect to a mean value, and the Price theorem relates the derivative of a mean with respect to a covariance, to means of first and second partials of the function whose average is being evaluated.

Bonnet's Theorem [7]

$$\partial E(f)/\partial\bar{z}_i = E(\partial f/\partial z_i) \quad (14a)$$

$$\partial E(f)/\partial\bar{z}_i^* = E(\partial f/\partial z_i^*). \quad (14b)$$

Price's Theorem [8]

$$\partial E(f)/\partial\Lambda_{ij} = 2E(\partial^2 f/\partial z_i\,\partial z_j^*). \quad (15)$$

These theorems are proved for complex Gaussian noise in exactly similar ways to ordinary Gaussian noise, by expressing the mean using the characteristic function and integrating by parts.

The moments of Gaussian noises may be computed by taking partial derivatives of the characteristic function and then setting the variables equal to zero. Thus we have

Miller's Formula [9]

$$E(z_1^{*n_1}\cdots z_k^{*n_k}z_1^{m_1}\cdots z_k^{m_k}) = (-2j)^{n_1+\cdots+m_k}(\partial/\partial\rho_1)^{n_1}\cdots(\partial/\partial\rho_k^*)^{m_k}\psi|_{\rho_1=\cdots=\rho_k^*=0}. \quad (16)$$

Reed [10] has given an explicit formula for the moments,

$$E[(z_{i_1} - \bar{z}_{i_1})^*\cdots(z_{i_n} - \bar{z}_{i_n})^*(z_{j_1} - \bar{z}_{j_1})\cdots(z_{j_m} - \bar{z}_{j_m})] = 0 \quad (17a)$$

if $m \neq n$, where i_k and j_k are integers from the set $(1,2,\cdots,)$; whereas if $m = n$

$$E[(z_{i_1} - \bar{z}_{i_1})^*\cdots(z_{i_n} - \bar{z}_{i_n})^*(z_{j_1} - \bar{z}_{j_1})\cdots(z_{j_n} - \bar{z}_{j_n})] = 2^n\sum\Lambda_{j_1,i_{\pi(1)}}\cdots\Lambda_{j_n,i_{\pi(n)}}, \quad \text{all permutations} \quad (17b)$$

where π is a permutation of the set of integers $(1,2,\cdots,n)$. Thus for a zero-mean process,

$$E(z_1^*z_2^*z_3z_4) = 4(\Lambda_{31}\Lambda_{42} + \Lambda_{32}\Lambda_{41}).$$

Not all complex Gaussian noise is circularly complex, e.g., a carrier AM modulated with Gaussian noise.

II. Univariate Complex Hermite Polynomials

We define the Hermite functions by the integral [4]

$$H_{nm}(z,z^*) = 2\pi\exp(zz^*/2)\int_{-\infty}^{\infty} du/2\pi\int_{-\infty}^{\infty} dv/2\pi \times (j\rho^*)^n(j\rho)^m\exp[-j(\rho^*z + z^*\rho)/2 - \rho\rho^*/2], \quad n - m = \text{integer}. \quad (18)$$

Because of the quadratic in the exponent, this integral is defined whether or not z^* is really the complex conjugate of z. Helstrom [16] mentions this. If the notation of having z^* the complex conjugate of z is followed, we will omit explicit representation of the variables or use the notation $H_{nm}(z)$ alone. If z^* is not the complex conjugate of z, then we indicate both arguments explicitly, as in the definition. Thus we will refer to the Hermite functions $H_{nm}(z,-z^*)$ and to $H_{nm}(jz,jz^*)$, and the second argument is clearly not the conjugate of the first. We always have $n - m$ an integer in our work.

By means of a simple change of variable, we have the following alternative representation

$$H_{nm}(z,z^*) = 2\pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du\,dv(j\rho^* + z^*)^n(j\rho + z)^m \exp(-\rho\rho^*/2)/(2\pi)^2. \quad (19)$$

This representation is valid for n and m positive integers.

The Hermite functions admit the following representation.

$$H_{nm}(z,z^*) = \frac{n!}{(n-m)!}z^{*(n-m)}(-2)^m \times M(-m, n-m+1, zz^*/2), \quad n \geq m, n \geq 0 \quad (20a)$$

$$H_{nm}(z,z^*) = \frac{m!}{(m-n)!}z^{(m-n)}(-2)^n \times M(-n, m-n+1, zz^*/2), \quad m \geq n, m \geq 0. \quad (20b)$$

This is derived by changing to polar coordinates and using the infinite integral formulas in Watson [6]. M is the confluent hypergeometric function ${}_1F_1$. Thus, the Hermite functions are proportional to Laguerre functions.

We can derive a recurrence formula for the Hermite functions by differentiating the integral (18). These formulas are not tabulated because they are so easily derived.

When n and m are positive integers, then we achieve the complex Hermite polynomials. These polynomials may be determined by a Rodrigues formula

$$H_{nm}(z,z^*) = (-2)^{n+m} \exp(zz^*/2)(\partial/\partial z)^n(\partial/\partial z^*)^m \exp(-zz^*/2). \quad (21)$$

The generating function is

$$\exp[(az^* + a^*z - aa^*)/2] = \sum \frac{(a/2)^n(a^*/2)^m}{n!\, m!} H_{nm}(z,z^*). \quad (22)$$

Another recurrence formula may be derived by differentiating this generating function.

The Hermite polynomials satisfy an orthogonality relationship.

$$\iint_{-\infty}^{\infty} dx\, dy \exp(-zz^*/2)H_{nm}(z)H_{\nu\mu}^*(z) = 2\pi\delta_{n\nu}\delta_{m\mu}\, n!\, m!\, 2^{n+m}. \quad (23)$$

The Mehler expansion for the bivariate Gaussian distribution constitutes a bivariate generating function for the complex Hermite polynomials

$$\exp[-(z_1z_1^* + z_2z_2^* - \Lambda_{21}z_1z_2^* - \Lambda_{12}z_2z_1^*)/2(1 - \Lambda_{12}\Lambda_{21})]/(2\pi)^2(1 - \Lambda_{12}\Lambda_{21})$$
$$= 1/(2\pi)^2 \sum_{n,m=0}^{\infty} \Lambda_{12}{}^n\Lambda_{21}{}^m H_{nm}(z_1)H_{mn}(z_2)/(2^{n+m}\, n!\, m!) \exp[-(z_1z_1^* + z_2z_2^*)/2]. \quad (24)$$

For Gaussian noise with mean $\bar{z}$ and variance σ^2, the Hermite polynomials have the following mean values.

$$E\{H_{nm}(\lambda z/\sigma)\} = (1 - |\lambda|^2)^{(n+m)/2}H_{nm}(\lambda\bar{z}/\sigma(1 - |\lambda|^2)^{1/2}). \quad (25)$$

Particular cases using this identity are obtained by setting λ equal to infinity and unity and using the limiting values for the Hermite polynomials for large values of the arguments. Thus we have the following result for the mean of the variable

$$E(z^{*n}z^m) = (-j\sigma)^{n+m}H_{nm}(j\bar{z}/\sigma, j\bar{z}^*/\sigma). \quad (26)$$

This holds for arbitrary n and m since it is derivable from the integral definition of the Hermite functions. The mean of the Hermite polynomial itself is

$$E(H_{nm}(z/\sigma)) = (\bar{z}^*/\sigma)^n(\bar{z}/\sigma)^m. \quad (27)$$

The limiting behavior is that as zz^* approaches infinity, the Hermite function approaches $z^{*n}z^m$. For small z we have H_{nm} approaching $(-2)^m z^{*(n-m)} n!/(n - m)!$ if $n \geq m$ whereas it approaches $(-2)^n z^{(m-n)} m!/(m - n)!$ if $m \geq n$. (More can be obtained from further examination of the behavior of the confluent hypergeometric function.)

Nonpositive integral values of n and m may be considered. Thus we have the following result

$$H_{n,-1}(z,z^*) = 2^n n!/z^{n+1}[e_n(zz^*/2) - \exp(zz^*/2)]$$

where the function

$$e_n(x) = 1 + x + \cdots + x^n/n!$$

is the truncated exponential function, and n is a nonnegative integer. In particular,

$$H_{0,-1} = (1 - \exp(zz^*/2))/z.$$

The complex Hermite polynomials may be related to the ordinary Hermite polynomials $He_k(x)$ by the following relation obtained from the generating function

$$2^k He_k[(z + z^*)/2] = \sum_{n=0}^{k} \binom{k}{n} H_{n,k-n}(z,z^*). \quad (28)$$

We define the complex Grad polynomials H by the formula

$$H_{(i,j)}{}^{(n,m)} = z_{i_1}^* \cdots z_{i_n}^* z_{j_1} \cdots z_{j_m} - 2(z_{i_1}^* \cdots z_{i_{n-1}}^* z_{j_1} \cdots z_{j_{m-1}}\delta_{i_n,j_m} + \cdots), \quad \binom{n}{1}\binom{m}{1} \text{ terms}$$
$$+ 4(z_{i_1}^* \cdots z_{i_{n-2}}^* z_{j_1}^* \cdots \times z_{j_{m-2}}\delta_{i_n,j_m}\delta_{i_{n-1},j_{m-1}} + \cdots), \quad \binom{n}{2}\binom{m}{2} \text{ terms}$$
$$- 8(\qquad), \text{ etc.},$$
$$i = (i_1,\cdots,i_n),\ j = (j_1,\cdots,j_m). \quad (29)$$

These occur in the complex Volterra series representation for nonlinear functionals. All the properties [11] of Grad polynomials may be extended to the complex Grad polynomials.

In this section, we have introduced the Hermite functions. They arise in several natural ways by means of power of z and z^*, the Mehler expansion, etc. These functions are proportional to confluent hypergeometric functions. The use of the notation H_{nm} yields formulas analogous to the usual formulas for Hermite polynomials, and usually with more simplicity than if the $M(\quad,\quad)$ notation is used.

III. Expansions in Hermite Polynomials

It is useful to consider expansions of functions in Hermite polynomials, because the Hermite polynomials are orthogonal. (In this section, we consider noise with zero mean and unit variance.) In the expansion,

$$f(z,z^*) \sim \sum a_{mn}H_{nm}(z), \quad (30)$$

that choice of coefficients that minimizes the mean-square error is

$$a_{mn} = \iint_{-\infty}^{\infty} \exp(-zz^*/2)f(z,z^*)H_{mn}(z)\, dx\, dy/(2\pi\, n!\, m!\, 2^{n+m}). \quad (31)$$

Suppose the function f, when expressed in terms of R and θ [$z = R\exp(j\theta)$] is $R^k g(\theta)$, and assume that g_n is defined by

$$g_n = \int_0^{2\pi} g(\theta) \exp(-jn\theta)\, d\theta/2\pi. \tag{32}$$

Then, if n is an integer larger than m, we have

$$a_{mn} = (-1)^m g_{m-n} 2^{(k-m-n)/2} \times \frac{\Gamma\left(\frac{k+n-m}{2}+1\right)\Gamma\left(\frac{m+n-k}{2}\right)}{n!\, m!\, \Gamma\left(\frac{n-m-k}{2}\right)}, \qquad n \geq m. \tag{33a}$$

This formula may be found by expressing the Hermite polynomials as a confluent hypergeometric function, and then using the formula [12, 6.10.6] for the Laplace transform of the product of a hypergeometric function and a power of the argument. The alternate result is

$$a_{mn} = (-1)^n g_{m-n} 2^{(k-n-m)/2} \times \frac{\Gamma\left(\frac{k+m-n}{2}+1\right)\Gamma\left(\frac{m+n-k}{2}\right)}{n!\, m!\, \Gamma\left(\frac{m-n-k}{2}\right)}, \qquad m \geq n. \tag{33b}$$

Examples

1) $z = R \exp(j\theta)$.

In this case $g_n = \delta_{n,1}$. Thus the only nonzero terms occur when $m - n = 1$ and here we have, according to the above formula,

$$a_{n+1,n} = (-1)^n 2^{-n} \Gamma(2)\Gamma(n)/\Gamma(0) n!\,(n+1)!.$$

Now $\Gamma(0)$ is infinite, and so the coefficients vanish except when n is zero, in which case we have $a_{1,0} = 1$. Thus $z = H_{0,1}$.

2) $1/|z| = R^{-1}$.

Here we have k equal to -1 and g_n vanishing except for n equal to zero. On application of the above formulas, we have the representation

$$1/|z| = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n(\frac{1}{2})_n}{n!\, n!} H_{nn}(z). \tag{34}$$

3) $z/|z| = \exp(j\theta)$.

Here $k = 0$, and g_n vanishes except when n is 1. This leads to the representation

$$z/|z| = \tfrac{1}{2}\sqrt{\pi/2} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n(\frac{1}{2})_n}{n!\,(n+1)!} H_{n,n+1}(z). \tag{35}$$

4) $1/z = R^{-1} \exp(-j\theta)$.

In this example, g_n vanishes except when n is equal to -1, and so we have $m - n = 1$. Thus we have the representation

$$1/z = \tfrac{1}{2} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n}{(n+1)!} H_{n+1,n}(z). \tag{36}$$

5) $|z| = R$.

Application of the above rules leads to

$$|z| = -\sqrt{2}/4 \sum_{n=0}^{\infty} \frac{\Gamma(n-\frac{1}{2})}{n!\, n!} (-\tfrac{1}{2})^n H_{nn}(z). \tag{37}$$

6) $\exp(-|z|s)/|z|$.

This example has been carefully chosen to exploit a formula of Erdélyi [12], thus,

$$\exp(-|z|s)/|z| = 1/\sqrt{2} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n}{n!} U(\tfrac{1}{2}, \tfrac{1}{2} - n; s^2/2) H_{nn}(z). \tag{38}$$

The function U is the other confluent hypergeometric function. This result agrees with that of 2) by considering the behavior of the coefficients when s becomes small.

7) $\exp(-s|z|^2)$.

This has the representation

$$\exp(-s|z|^2) = \sum_{n=0}^{\infty} \left(\frac{-s/2}{s+2}\right)^n \frac{1}{n!} \frac{H_{nn}(z)}{(1+s/2)}, \qquad s \geq 0,\ s \text{ real}. \tag{39}$$

Another common technique to derive expansions in Hermite polynomials is to use integration by parts, that is, to use the formula for a_{mn} in the form

$$a_{mn} = (-1)^{n+m} \int_{-\infty}^{\infty} f(z)(\partial/\partial z)^m(\partial/\partial z^*)^n \exp(-zz^*/2)\, dx\, dy/2\pi\, n!\, m!. \tag{40}$$

But this does not appear to be as useful as one might expect.

There is, of course, the obvious representation for the impulse functions given by

$$\delta(x)\delta(y) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n}{n!} H_{nn}(z). \tag{41}$$

It is possible to give an alternative derivation for these formulas that is sometimes useful. It makes use of the formula

$$\int_{-\infty}^{\infty} H_{nm}(z) \exp(-|z - \bar{z}|^2/2)\, dx\, dy/2\pi = \bar{z}^m \bar{z}^{*n}. \tag{42}$$

To derive the expansions indicated above, form the expectation of the function $f(z)$ where z is a unit variance Gaussian noise with nonzero mean $\bar{z}$; then expand the expectation as a power series in $\bar{z}$ and $\bar{z}^*$; then to determine the expansion of $f(z)$ replace $\bar{z}^{*n}\bar{z}^m$ with $H_{nm}(z)$. This observation allows the straightforward derivation of the following expansions,

$$z^{*n}z^m = \frac{n!}{(n-m)!} 2^m \sum \frac{(-m)_k(-\frac{1}{2})^k}{(n-m+1)_k\, k!} H_{n-m+k,k}(z,z^*), \qquad n \geq m,\ n \geq 0 \tag{43a}$$

and

$$z^{*n}z^m = \frac{m!}{(m-n)!} 2^n \sum_{k=0}^{\infty} \frac{(-n)_k(-\frac{1}{2})^k}{(m-n+1)_k\, k!} H_{k,m-n+k}(z,z^*), \qquad m \geq n,\ m \geq 0. \tag{43b}$$

If we have an expansion in Hermite polynomials then the computation of the mean is elementary using (27).

The probability density of the phase $p(\theta)$ is periodic in θ with period 2π, and hence has a Fourier representation $\sum a_k \exp(jk\theta)$. The coefficient a_k is given by

$$a_k = \int_0^{2\pi} \exp(-jk\theta)p(\theta)\, d\theta/2\pi = E(z^{*k/2}z^{-k/2})$$

and representation of $H_{k/2,-k/2}(j\bar{z}/\sigma, j\bar{z}^*/\sigma)$ yields the formula

$$p(\theta) = \sum H_{k/2,-k/2}(j\bar{z}/\sigma, j\bar{z}^*/\sigma) \exp(jk\theta). \quad (44)$$

This appears in Middleton [3].

IV. Bivariate Complex Hermite Polynomials

In many computations in noise theory, it is required to compute second moments, and for this purpose we must consider the bivariate complex Hermite polynomials. Although it is possible to generalize to n-variate polynomials following a program laid out by Appell and Kampé de Feriet [14], we have found it more useful to consider only the bivariate polynomials in detail.

The bivariate Hermite polynomial is defined by the integral

$$H_{n,m}^{(2)}(z,z^*) = (2\pi)^2 \exp(z^\dagger \Lambda^{-1} z/2) \det \Lambda \cdot \int_{-\infty}^{\infty}\cdots\int du_1\, dv_1\, du_2\, dv_2 (j\rho_1^*)^{n_1}(j\rho_2^*)^{n_2} \times (j\rho_1)^{m_1}(j\rho_2)^{m_2} \exp[-j(\rho^\dagger z + z^\dagger \rho)/2 - \rho^\dagger \Lambda \rho/2]. \quad (45)$$

where

$$n = (n_1, n_2)$$
$$m = (m_1, m_2)$$
$$z = (z_1, z_2)$$
$$z^* = (z_1^*, z_2^*)$$
$$\rho = (u_1 + jv_1, u_2 + jv_2)$$
$$n_1 - m_1 = \text{integer}$$
$$n_2 - m_2 = \text{integer}.$$

The matrix Λ in the exponent is the second-order covariance matrix. The Hermite polynomial is a function of eight variables in all, $z_1, z_2, z_1^*, z_2^*, \Lambda_{11}, \Lambda_{22}, \Lambda_{12}, \Lambda_{21}$. As before, we treat z and z^* in this integral as completely unrelated complex numbers and indicate this with explicit representation, as in $H_{n,m}^{(2)}(z,z^*)$. If this is not the case, and z^* is indeed the conjugate of z, this is indicated by $H_{n,m}^{(2)}(z)$ or $H^{(2)}(z)$.

The presentation of the results for this section will parallel that for the univariate Hermite polynomials, with the exception of the necessity to introduce the G polynomials.

In dealing with the bivariate Hermite polynomials, it is useful to consider the partials of the quadratic form with respect to the variables z. Thus we define

$$w_1 = (\partial/\partial z_1^*) z^\dagger \Lambda^{-1} z = (\Lambda_{22} z_1 - \Lambda_{12} z_2)/\det \Lambda$$
$$w_1^* = (\partial/\partial z_1) z^\dagger \Lambda^{-1} z \quad (46)$$

and, in general, $w = \Lambda^{-1} z$, where $z = \Lambda w$ and

$$z^\dagger \Lambda^{-1} z = z^\dagger w = w^\dagger z = w^\dagger \Lambda w. \quad (47)$$

The numbers z may be similarly defined by taking partials with respect to w. Thus,

$$z_1 = (\partial/\partial w_1^*)(w^\dagger \Lambda w).$$

We are now in a position to define the polynomial G by the formula

$$G_{n,m}^{(2)}(z,z^*) = (2\pi)^2 \exp(w^\dagger \Lambda w/2) \det \Lambda^{-1} \cdot \int_{-\infty}^{\infty}\cdots\int du_1\, dv_1\, du_2\, dv_2/(2\pi)^4 (j\rho_1^*)^{n_1} \times (j\rho_2^*)^{n_2}(j\rho_1)^{m_1}(j\rho_2)^{m_2} \times \exp[-j(\rho^\dagger w + w^\dagger \rho)/2 - \rho^\dagger \Lambda^{-1} \rho/2],$$
$$n_1 - m_1 = \text{integer},\ n_2 - m_2 = \text{integer},\ \rho_1 = u_1 + jv_1, \text{ etc.} \quad (45b)$$

The G polynomial is the same as an H polynomial except that z and w are interchanged, as are Λ and Λ^{-1}.

The H polynomials allow the following alternative representation

$$H_{n,m}^{(2)}(z,z^*) = (2\pi)^2 \det \Lambda \int_{-\infty}^{\infty}\cdots\int du_1\, dv_1\, du_2\, dv_2/(2\pi)^4 \times (j\rho_1^* + w_1^*)^{n_1}(j\rho_2^* + w_2^*)^{n_2} \times (j\rho_1 + w_1)^{m_1}(j\rho_2 + w_2)^{m_2} \exp(-\rho^\dagger \Lambda \rho/2),$$
$$n_1, m_1, n_2, m_2 \text{ positive integers.} \quad (48)$$

The Rodrigues formula for the Hermite polynomials H is

$$H_{n,m}^{(2)}(z,z^*) = \exp(z^\dagger \Lambda^{-1} z/2)(-2\, \partial/\partial z_1)^{n_1} \cdots \times (-2\, \partial/\partial z_2^*)^{m_2} \exp(-z^\dagger \Lambda^{-1} z/2). \quad (49)$$

The generating function is

$$\exp\{[-(z-a)^\dagger \Lambda^{-1}(z-a)/2] + z^\dagger \Lambda^{-1} z/2\}$$
$$= \exp[(a^\dagger w + w^\dagger a - a^\dagger \Lambda^{-1} a)/2]$$
$$= \sum \frac{(a_1/2)^{n_1}(a_1^*/2)^{m_1}(a_2/2)^{n_2}(a_2^*/2)^{m_1}}{n_1!\, m_1!\, n_2!\, m_2!} H_{n,m}^{(2)}(z,z^*). \quad (50)$$

Analogous formulas may be written for the polynomials G.

It is possible to compute recurrence formulas by differentiating the defining integral with respect to the variables z_i and z_i^*, and also by differentiating the generating function with respect to the variables a_1, etc.

The orthogonality formula is more complicated than that for the univariate Hermite polynomials in that it involves both the H and the G polynomials. This orthogonality formula is

$$\int_{-\infty}^{\infty} dx_1\, dx_2\, dy_1\, dy_2 \exp(-z^\dagger \Lambda^{-1} z/2) H_{n,m}^{(2)}(z) \times G_{\nu,\mu}^{(2)*}(z)/(2\pi)^2 \det \Lambda = \delta_{n_1,\nu_1}\delta_{m_1,\mu_1}\delta_{n_2,\nu_2}\delta_{m_2,\mu_2} 2^{n_1+m_1+n_2+m_2} n_1!\, n_2!\, m_1!\, m_2!. \quad (51)$$

It may be derived in the usual way, integration by parts. It is necessary to use the result that the polynomial H is of degree n_1 in w_1^*, m_1 in w_1, n_2 in w_2^*, and m_2 in w_2 and that the polynomial G is of degree n_1 in z_1^*, m_1 in z_1, n_2 in z_2^*, and m_2 in z_2.

The Mehler expansion (quadrivariate generating function) is determined in the following way. Suppose that the 4×4 matrix Λ be decomposed into the 2×2 matrices Λ_a and Λ_b, and that the off-diagonal matrices have their diagonal terms vanishing, as indicated by the chart.

$$\Lambda = \left[\begin{array}{c|c} \Lambda_a = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} & \begin{matrix} 0 & \Lambda_{14} \\ \Lambda_{23} & 0 \end{matrix} \\ \hline \begin{matrix} 0 & \Lambda_{32} \\ \Lambda_{41} & 0 \end{matrix} & \Lambda_b = \begin{bmatrix} \Lambda_{33} & \Lambda_{34} \\ \Lambda_{43} & \Lambda_{44} \end{bmatrix} \end{array}\right]$$

The density function corresponding to Gaussian noise is then expanded with this covariance matrix in terms of the coefficients Λ_{14}, Λ_{23}, Λ_{32}, and Λ_{41}. The result is the Mehler expansion,

$$\begin{aligned} &\exp(-z^\dagger \Lambda^{-1} z/2)/(2\pi)^4 \det \Lambda \\ &\quad = \exp(-z_a^\dagger \Lambda_a^{-1} z_a/2)/(2\pi)^2 \det \Lambda_a \\ &\quad \times \exp(-z_b^\dagger \Lambda_b^{-1} z_b/2)/(2\pi)^2 \det \Lambda_b \\ &\quad \cdot \sum \frac{(\Lambda_{14}/2)^{m_1}(\Lambda_{41}/2)^{n_1}(\Lambda_{23}/2)^{m_2}(\Lambda_{32}/2)^{n_2}}{m_1!\, n_1!\, m_2!\, n_2!} \\ &\quad \times H_{n_1,n_2;m_1,m_2}^{(2)}(z_a,z_a^*) H_{m_2,m_1;n_2,n_1}^{(2)}(z_b,z_b^*). \quad (52) \end{aligned}$$

For Gaussian noise with mean $\bar{z}$ and covariance Λ, the mean value of the Hermite polynomial $H^{(2)}$ is determined by the formula

$$\begin{aligned} &\int_{-\infty}^{\infty} H_{n,m}^{(2)}(\lambda z) \exp[-(z-\bar{z})^\dagger \times \Lambda^{-1}(z-\bar{z})/2]\, dx_1\, dx_2\, dy_1\, dy_2/(2\pi)^4 \det \Lambda \\ &\quad = (1-\lambda^2)^{(n_1+m_1+n_2+m_2)/2} \times H_{n,m}^{(2)}(\lambda\bar{z}/\sqrt{1-\lambda^2}, \lambda\bar{z}^*/\sqrt{1-\lambda^2}) \quad (53a) \end{aligned}$$

and the mean value of the G polynomial is

$$\begin{aligned} &\int_{-\infty}^{\infty} G_{n,m}^{(2)}(\lambda z) \exp[-(z-\bar{z})^\dagger \times \Lambda^{-1}(z-\bar{z})/2]\, dx_1\, dx_2\, dy_1\, dy_2/(2\pi)^4 \det \Lambda \\ &\quad = (1-\lambda^2)^{(n_1+m_1+n_2+m_2)/2} \times G_{n,m}^{(2)}(\lambda\bar{z}/\sqrt{1-\lambda^2}, \lambda\bar{z}^*/\sqrt{1-\lambda^2}). \quad (53b) \end{aligned}$$

The four results that are obtained by setting λ equal to 1 and ∞ are

$$E\{H_{n,m}^{(2)}(z)\} = \bar{w}_1^{*n_1}\bar{w}_2^{*n_2}\bar{w}_1^{m_1}\bar{w}_2^{m_2} \quad (54a)$$

$$E\{w_1^{*n_1}w_2^{*n_2}w_1^{m_1}w_2^{m_2}\} = j^{-(n_1+m_1+n_2+m_2)} H_{n,m}^{(2)}(j\bar{z}, j\bar{z}^*) \quad (55a)$$

$$E\{G_{n,m}^{(2)}(z)\} = \bar{z}_1^{*n_1}\bar{z}_2^{*n_2}\bar{z}_1^{m_1}\bar{z}_2^{m_2} \quad (54b)$$

$$E\{z_1^{*n_1}z_2^{*n_2}z_1^{m_1}z_2^{m_2}\} = j^{-(n_1+m_1+n_2+m_2)} G_{n,m}^{(2)}(j\bar{z}, j\bar{z}^*) \quad (55b)$$

Equations (55a) and (55b) are valid for arbitrary n_1 etc. as long as the limitation that $n_1 - m_1$ and $n_2 - m_2$ be integers is maintained, since they may be deduced from the defining integrals for the functions H and G.

Erdélyi [12] has indicated how to express ordinary Hermite polynomials in terms of the Hermite polynomials; the result is

$$\begin{aligned} &He_k[(z^\dagger \Lambda^{-1} a + a^\dagger \Lambda^{-1} z)/(2\sqrt{a^\dagger \Lambda^{-1} a})] \\ &\quad = k!/(a^\dagger \Lambda^{-1} a)^{k/2} 2^k \sum_{n_1+m_1+n_2+m_2=k} \frac{a_1^{n_1} a_2^{n_2} a_1^{*m_1} a_2^{*m_2}}{n_1!\, n_2!\, m_1!\, m_2!} H_{n,m}^{(2)}(z). \quad (56) \end{aligned}$$

This is derived by setting $a = ta$ in the generating formula, where t is a real number, and then using the generating formula for the Hermite polynomials. We may also express the univariate complex Hermite polynomials in terms of the bivariate polynomials by the substitution $a = ua$, with u complex; the result is

$$\begin{aligned} &H_{n,m}(a^\dagger w, w^\dagger a) \\ &\quad = (a^\dagger \Lambda^{-1} a)^{-(m+n)/2} \sum \binom{n}{r}\binom{m}{j} a_1^{n-r} a_1^{*m-s} a_2^{r} a_2^{*s} \times H_{n-r,r;m-s,s}^{(2)}(z,z^*). \quad (57) \end{aligned}$$

It is straightforward to derive analogous formulas for the G polynomial.

The Taylor series for the G polynomial is

$$\begin{aligned} &G_{n,m}^{(2)}(z+\delta z, z^* + \delta z^*) \\ &\quad = \sum_{r_1,r_2,s_1,s_2=0}^{\infty} \frac{\delta z_1^{*r_1}\delta z_1^{s_1}\delta z_2^{*r_2}\delta z_2^{s_2}}{r_1!\, s_1!\, r_2!\, s_2!} (-1)^{r_1+s_1+r_2+s_2} \\ &\quad \times (-n_1)_{r_1}(-n_2)_{r_2}(-m_1)_{s_1}(-m_2)_{s_2} \\ &\quad \times G_{n_1-r_1,n_2-r_2;m_1-s_1,m_2-s_2}^{(2)}(z,z^*) \quad (58) \end{aligned}$$

and this may be derived by using the recurrence formula for the G polynomial derived by taking a derivative of the expression analogous to (51) for the G polynomial.

The analogous formula for the H polynomial is

$$\begin{aligned} &H_{n,m}^{(2)}(w+\delta w, w^* + \delta w^*) \\ &\quad = \sum \frac{\delta w_1^{*r_1}\delta w_1^{s_1}\delta w_2^{*r_2}\delta w_2^{s_2}}{r_1!\, s_1!\, r_2!\, s_2!} (-1)^{r_1+s_1+r_2+s_2} \\ &\quad \times (-n_1)_{r_1}(-n_2)_{r_2}(-m_1)_{s_1}(-m_2)_{s_2} \\ &\quad \times H_{n_1-r_1,n_2-r_2;m_1-s_1,m_1-s_2}^{(2)}(w,w^*). \quad (59) \end{aligned}$$

The peculiarity of this formula is that the diminution of order of the H polynomials is effected by taking partials with respect to w and not z.

It is possible to derive some formulas for the complex Hermite polynomials in terms of simpler functions by expanding the factors in the exponent of the defining integral. Expansion in the covariances Λ_{12} and Λ_{21} leads to the formula

$$H_{n,m}^{(2)}(z,z^*) = \sum_{r,s=0}^{\infty} \frac{(\Lambda_{12}/2\sigma_1\sigma_2)^r}{r!} \frac{(\Lambda_{21}/2\sigma_1\sigma_2)^s}{s!} \times H_{n_1+r,m_1+s}^{(1)}(z_1/\sigma_1,z_1^*/\sigma_1) H_{n_2+s,m_2+r}(z_2/\sigma_2,z_2^*/\sigma_2) \times \exp(z^\dagger\Lambda^{-1}z/2 - z_1z_1^*/2\Lambda_{11} - z_2z_2^*/2\Lambda_{22}) \times \det \Lambda/\sigma_1^{m_1+n_1}\sigma_2^{m_2+n_2}, \quad \sigma_1^2 = \Lambda_{11}, \sigma_2^2 = \Lambda_{22}. \tag{60}$$

Expansions in the variables z_1, etc. yields the representations

$$G_{n,m}^{(2)}(z) = \sum \frac{z_1^{*r_1} z_1^{s_1} z_2^{*r_2} z_2^{s_2}}{r_1!\, s_1!\, r_2!\, s_2!} (-n_1)_{r_1}(-m_1)_{s_1}(-n_2)_{r_2} \times (-m_2)_{s_2} G_{n_1-r_1,n_2-r_2;m_1-s_1,m_2-s_2}^{(2)}(0). \tag{61}$$

$$H_{n,m}^{(2)}(z,z^*) = \sum \frac{w_1^{*r_1} w_1^{s_1} w_2^{*r_2} w_2^{s_2}}{r_1!\, s_1!\, r_2!\, s_2!} (-n_1)_{r_1}(-m_1)_{s_1}(-n_2)_{r_2}(-m_2)_{s_2} \times H_{n_1-r_1,n_2-r_2;m_1-s_1,m_2-s_2}^{(2)}(0). \tag{62}$$

Reed [5] has computed the bivariate Hermite polynomials when the argument z vanishes. His result, modified to our notation, follows.

If $n_1 - m_1 = r > 0$ then

$$H_{n,m}^{(2)}(0) = \delta_{n_1-m_1,m_2-n_2}(-2)^{n_1+m_2}(\Lambda_{21}/2\Lambda_{11}\Lambda_{22})^{n_1-m_1} \times \frac{n_1!\, m_2!}{(n_1-m_1)!} \frac{\det \Lambda}{\Lambda_{11}\Lambda_{22}} \Lambda_{11}^{-m_1}\Lambda_{22}^{-n_2} \times {}_2F_1(n_1+1, m_2+1; n_1-m_1+1; \Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22}). \tag{63a}$$

If $n_1 - m_1 = r > 0$ then

$$G_{n,m}^{(2)}(0) = \delta_{n_1-m_1,m_2-n_2}(-2)^{m_1+m_2}(\Lambda_{21}/\Lambda_{11}\Lambda_{22})^{n_1-m_1} \times \frac{n_1!\, m_2!}{(n_1-m_1)!} \Lambda_{11}^{m_2}\Lambda_{22}^{n_1}\, {}_2F_1(-m_1, -n_2; n_1-m_1+1; \Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22}). \tag{63b}$$

If $m_1 - n_1 = r > 0$ then

$$H_{n,m}^{(2)}(0) = \delta_{m_1-n_1,n_2-m_2}(-2)^{m_1+n_2}(\Lambda_{12}/2\Lambda_{11}\Lambda_{22})^{m_1-n_1} \times \frac{n_2!\, m_1!}{(m_1-n_1)!} \frac{\det \Lambda}{\Lambda_{11}\Lambda_{22}} \Lambda_{11}^{-n_1}\Lambda_{22}^{-m_2} \times {}_2F_1(m_1+1, n_2+1; m_1-n_1+1; \Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22}). \tag{64a}$$

If $m_1 - n_1 = r > 0$ then

$$G_{n,m}^{(2)}(0) = \delta_{m_1-n_1,n_2-m_2}(-2)^{n_1+n_2}(\Lambda_{12}/\Lambda_{11}\Lambda_{22})^{m_1-n_1} \times \frac{n_2!\, m_1!}{(m_1-n_1)!} \Lambda_{11}^{n_2}\Lambda_{22}^{m_1}\, {}_2F_1(-n_1, -m_2; m_1-n_1+1; \Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22}). \tag{64b}$$

For example

$$E(1/z_1^*z_2) = -/2\Lambda_{21} \ln(1 - \Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22}) = G_{-1,0;0,-1}^{(2)}(0) = -\tfrac{1}{2}(\Lambda_{12}/\Lambda_{11}\Lambda_{22})\, {}_2F_1(1,1;2;\Lambda_{12}\Lambda_{21}/\Lambda_{11}\Lambda_{22})$$

for zero mean processes z_1,z_2.

The two expansions given, the expansion in the covariance leading to confluent hypergeometric functions and the Taylor-series expansion leading to hypergeometric functions, correspond to two of the cases given by Middleton for the computation of moments. Middleton gives a third expansion that we have not been able to extend to the more general problem considered in this paper [3, p. 416].

The bivariate complex Hermite polynomials may be extended to nonpositive-integer values of n_1 etc. in the same way as the extension is made for the univariate complex Hermite functions. Two of particular interest for FM problems are $H_{-1;0,0,-1}^{(2)}$ and $H_{-1,-1;0,0}^{(2)}$. The results are

$$H_{-1,-1;0,0}^{(2)} = 1/w_1^*w_2^* - \exp(w_1^*w_1 \det \Lambda/2\Lambda_{22})\Lambda_{22}/w_1^*z_2^* - \exp(w_2^*w_2 \det \Lambda/2\Lambda_{11})\Lambda_{11}/w_2^*z_1^* + \exp(w^\dagger\Lambda w/2) \det \Lambda/z_1^*z_2^* \tag{65}$$

and

$$H_{-1,0;0,-1}^{(2)} = -\exp(-w_2w_1^* \det \Lambda/2\Lambda_{21}) \det \Lambda/2\Lambda_{21} \times [-Ei(w_2w_1^* \det \Lambda/2\Lambda_{21}) + Ei(z_2w_1^* \det \Lambda/2\Lambda_{22}\Lambda_{11}) + Ei(z_1^*w_2 \det \Lambda/2\Lambda_{11}\Lambda_{21}) - Ei(z_2z_1^*/2\Lambda_{21})],$$

$$Ei(x) = \int_{-\infty}^{x} \exp(u)\, du/u. \tag{66}$$

The derivation of the first formula is performed by integration by parts. The second is derived by integration with respect to z_1 and z_2^* of

$$\partial^2 H_{1,0;0,-1}^{(2)} \exp(-z^\dagger\Lambda^{-1}z/2)/\partial z_1\, \partial z_2^*.$$

In this section we have presented the bivariate complex Hermite functions and shown their properties. Recurrence formulas may be obtained in the ways indicated. The Taylor series expansions using Reed's formula (useful for small signals) and the expansion in confluent hypergeometric functions (useful for small correlation coefficient) have been presented. In addition, explicit formulas for two special rational functions have been determined.

Recently a note by Bédard [15] showed the same type of generalized Hermite polynomials that are discussed here. However, his discussion is not for circularly complex noises explicitly, and so there is a doubling of the number of variables used. Also, his discussion is related to polynomials through the generating function and Rodrigues' formula, whereas we have used the integral definition because of its possibilities of generalization of the numbers n_1, etc. to numbers that are not positive integers.

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How Does a Porcupine Separate Its Quills?

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***Abstract*—Sets of unit vectors in N-dimensional Euclidean vector space whose constituent vectors are separated one from another by at least a fixed distance d, prescribed once for all and independent of N, are of interest in theory and practice; they have fondly been called "porcupine codes." Although an elegant constructive proof of Gilbert shows that the number of vectors in a porcupine code (of given d) can increase exponentially with N, no systematic method is yet known for generating porcupine codes of this cardinality.**

Corresponding to a collection of M vectors, we can partition the space into maximum-likelihood regions, the jth of which consists of those vectors that lie closer to the jth than to any other element of the collection. Each maximum-likelihood region is bounded by at most $(M - 1)$ hyperplanes, and we denote by K the total number of these bounding hyperplanes. Collections for which K is small may be expected to have greater symmetry than those for which K is large.

In this paper we show that, for porcupine codes, $K \geq (M/2)^{1/s}$, with s depending only on d, the minimum separation of the code vectors. Hence, for the number of vectors of a porcupine code to increase exponentially with dimension, the number of separating hyperplanes must do so as well. We conclude with, an application to the permutation codes introduced by Slepian, showing that the number of vectors of a porcupine code which is of permutation-modulation type can not increase exponentially with N.

Manuscript received May 11, 1970.
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I. Introduction

A. Porcupine Codes

In THE geometric view of coding theory for a time-continuous channel, a message of fixed energy is represented as a unit vector in a space whose dimension is proportional to the bandwidth of the communication system. The sender, with M possible messages to transmit, fixes a set of M distinct unit vectors, called a "code," to represent them; the code is assumed known to the receiver. However, as transmission entails error, the received vector corresponding to a particular sent message from this code set does not necessarily coincide with one of the code vectors agreed upon, so that the receiver must guess which message had been intended. We suppose the perturbations of transmission to be such that his best strategy consists of choosing that one of the possible sent vectors which lies nearest to the received vector; this procedure is termed "maximum-likelihood decoding." Knowledge of the decoding strategy in turn influences the choice of code; in particular, it suggests unit vectors separated one from another by at least a prescribed distance $d > 0$, fixed independently of dimension. Sets of such vectors owe to J. H. VanLint the fond name of "porcupine codes."