Accurate Error-Rate Calculations Through the Inversion of Mixed Characteristic Functions

Lisa Welburn, James K. Cavers, *Fellow, IEEE,* and Kevin W. Sowerby, *Member, IEEE*

***Abstract*—This letter presents a new computational tool for use in general fading channel analyses when the detection scheme can be expressed as a quadratic form in zero-mean complex Gaussian random variables. We develop a simple numerical algorithm which is capable of inverting a characteristic function consisting of both simple and multiple poles. The approach benefits from the inherent symmetry in the residue calculations and uses the well-known Vandermonde matrix in order to take advantage of this symmetry. It is numerically stable, eliminates singularities, and circumvents the need for differentiation.**

***Index Terms*—Cauchy's residue theorem, inverse Laplace transform (ILT), Vandermonde matrix.**

I. Introduction

WHEN CALCULATING an error rate or the probability of flagging an erasure in fading channels [1], most thresholding schemes can be reduced to the evaluation of a quadratic form in Gaussian variates. For the special case of Rayleigh fading, the complex Gaussian variates are zero mean, and the characteristic function of the thresholded variable f is given by [2]

$$\phi_f(s) = \frac{A}{\prod_{k=1}^{N_L}(s - p_{l_k})\prod_{m=1}^{N_R}(s - p_{r_m})} \tag{1}$$

where $A = \prod_{k=1}^{N_L} p_{l_k} \prod_{m=1}^{N_R} p_{r_m}, p_{l_k}$ are the N_L left-plane (LP) poles, p_{r_m} are the N_R right-plane (RP) poles, and the poles are not necessarily distinct.

In order to calculate the probability of error or of flagging an erasure, we can take the two-sided inverse Laplace transform (ILT) of $\phi_f(s)/s$ to obtain the cumulative distribution function (CDF) of the random variable, f. Through the use of Cauchy's residue theorem [5], the ILT can be determined by calculating the residues at the RP or LP poles as shown in [3].

Inversion of $\phi_f(s)/s$ by residues is straightforward if $\phi_f(s)$ contains simple poles only. However, if it contains multiple poles, as is common in diversity problems, residue calculation requires tedious differentiations. The varying signs of the residues and their small sum also produce numerical instability.

In [4], the generalized inverse Vandermonde matrix is used to invert the Laplace transfer function of a time-invariant system with both multiple and simple poles. Our situation is different from that in [4] in two ways. First, we are dealing with the two-sided Laplace transform as opposed to the one-sided Laplace transform, and second, the poles of our transfer function change with the fading channel statistics. Since differences in the fading channel statistics change the pole locations, we need a simple method for handling the situation where the pole multiplicity can change with the operating conditions.

This letter presents a simple new method for using residues to invert a characteristic function with both simple and multiple poles. It exploits symmetry in the residues, in order to use the properties of the Vandermonde matrix [5] to provide explicit cancellation of factors in the numerator and denominator, thereby avoiding the singularities and numerical instabilities that plague conventional inversion.

II. Residue Calculation Method

The CDF of f, $P(f < \psi)$, is determined by calculating the two-sided ILT of $\phi_f(s)/s$. Rather than keeping the additional pole at zero separate, $\phi_f(s)$ is modified so that this additional pole is included into the LP poles such that $p_{l_{NL+1}} = 0$. In order to describe the residue calculation method, it is assumed that $\psi \leq 0$, and the residues will be taken over the RP poles. The modification for $\psi > 0$ is straightforward.

Since it is assumed that $\psi \leq 0$, the RP poles, which are all real, are increasingly ordered and are represented as $p_{r_m} = p_r + \delta_m$ and $0 \leq \delta_1 \leq \delta_2 \leq, \ldots, \leq \delta_{N_R}$. The residues at the RP poles are given by

$$R_n = \frac{Ae^{(p_r+\delta_n)\psi}}{\prod_{k=1}^{N_L+1}(p_r - p_{l_k} + \delta_n)\prod_{\substack{m=1\\m\neq n}}^{N_R}(\delta_n - \delta_m)}. \tag{2}$$

If the RP poles are distinct, then (2) presents no problems. However, if the RP poles come closer together to become multiple poles, $(\delta_n - \delta_m)$ in (2) approaches zero, and some residues will contain singularities. However, we are interested only in the sum of the residues, which as shown below, leads to cancellation of all factors of the form $(\delta_n - \delta_m)$, thus eliminating the singularities.

In order to eliminate the terms in the denominator, each residue is multiplied by a common factor, $C = GD$,where

$$G = \prod_{m=1}^{N_R}\prod_{k=1}^{N_L+1}(p_{r_m} - p_{l_k}) \tag{3}$$

$$D = \prod_{i=1}^{N_R}\prod_{j=i+1}^{N_R}(\delta_j - \delta i). \tag{4}$$

The modified residues, denoted $C_n = R_n C$, are given by

$$C_n = Ae^{p_r\psi}\left((-1)^{N_R-n}B_n D_n\right) \tag{5}$$

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L. Welburn and J. K. Cavers are with the School of Engineering Science, Simon Fraser University, Burnaby, BC V5A 1S6, Canada (e-mail: cavers@sfu.ca).

K. W. Sowerby is with the Radio Systems Group, Department of Electrical and Electronic Engineering, University of Auckland, Auckland, New Zealand.

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where

$$B_n = e^{\delta_n \psi} \prod_{\substack{m=1 \\ m \neq n}}^{N_R} \prod_{k=1}^{N_L+1} (p_r - p_{l_k} + \delta_m) \tag{6}$$

is positive, and

$$D_n = \prod_{\substack{i=1 \\ i \neq n}}^{N_R} \prod_{\substack{j=i+1 \\ j \neq n}}^{N_R} (\delta_j - \delta_i) \tag{7}$$

is merely nonnegative. The CDF of f is given by

$$P(f < \psi) = \sum_{n=1}^{N_R} \frac{C_n}{C} = \frac{-Ae^{p_r \psi}}{C} \sum_{n=1}^{N_R} (-1)^{N_R - n} B_n D_n. \tag{8}$$

It can be shown that D_n and D are both determinants of Vandermonde matrices [5], where D_n represents D, but with the nth column and the last row removed. Therefore, from (8), it can be observed that the formulation of $P(f < \psi)$ is equivalent to evaluating the determinant of a matrix through the expansion of minors. In (8), D_n is the minor associated with B_n, and the CDF of f is

$$P(f < \psi) = \frac{-Ae^{p_r \psi} \det(\mathbf{X})}{C} \tag{9}$$

where

$$\mathbf{X} = \begin{bmatrix} \delta_1^0 & \delta_2^0 & \cdots & \delta_{N_R}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1^{N_R-2} & \delta_2^{N_R-2} & \cdots & \delta_{N_R}^{N_R-2} \\ B_1 & B_2 & \cdots & B_{N_R} \end{bmatrix}. \tag{10}$$

The divisor of (9) is $C = GD$. Because D can be very small or zero when $\delta_j - \delta_i \cong 0$, the division can cause numerical problems. We, therefore, factor D out of the determinant, giving $\det(\mathbf{X}) = D \det(\mathbf{H})$, and cancel D to obtain the numerically stable

$$P(f < \psi) = \frac{-Ae^{p_r \psi} \det(\mathbf{H})}{G}. \tag{11}$$

The matrix, $\mathbf{H}$, can be calculated in an iterative fashion by exploiting the following properties of determinants.

Property #1: To any column of the matrix we can add any multiple of any other column without changing the determinant.

Property #2: A common factor of all the elements in a row or a column can be taken outside the determinant.

Fig. 1 provides an example to show the iterative procedure for $\mathbf{H}$, and the formal derivation can be found in [6, App. A]. As can be seen in Fig. 1, $\mathbf{H}$ becomes a lower triangular matrix. Therefore, $\det(\mathbf{H}^{(3)})$ will simply be the product of the diagonal elements, a simple calculation since the only diagonal element not equal to one is $[\mathbf{H}^{(3)}]_{3,3}$. Proof of divisibility is given in [6, App. A].

III. Implementation

We now present an iterative procedure for calculating $\det(\mathbf{H})$ by factoring D from $\mathbf{X}$. It is numerically stable, requires no divisions or symbolic processing, and completes in $N_R - 1$ iterations. We first define

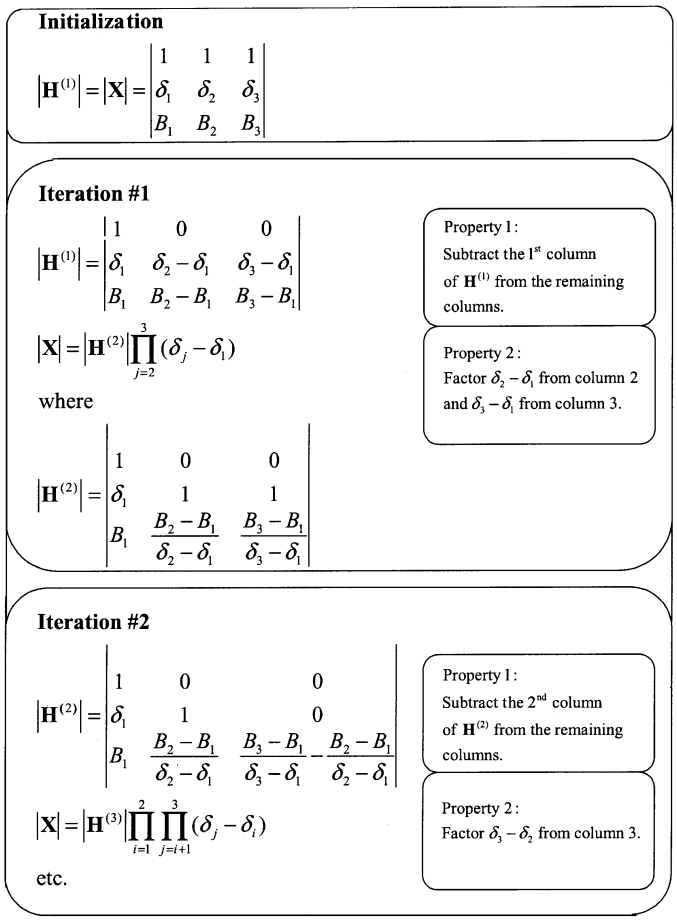

Fig. 1. Iterated matrix, $\mathbf{H}^{(i)}$, for three RP poles.

$$P_1(\delta_i) = \prod_{k=1}^{N_L+1} (p_r - p_{l_k} + \delta_i) = \sum_{w=0}^{N_L+1} a_w \delta_i^w \tag{12}$$

$$P_2^{(1)}(\delta_i) = e^{\delta_i \psi} = \sum_{w=0}^{L_b^{(1)}} b_w^{(1)} \delta_i^w \tag{13}$$

where $e^{\delta_i \psi}$ is expanded using the Maclaurin series, and it is truncated at $L_b^{(1)}$ terms. The arbitrary parameter p_r can be selected to minimize the length of the Maclaurin series. The number of terms depends on the accuracy desired, the threshold, and the pole spread. For example, with a threshold of $\psi = -10$ and a pole spread that corresponds to a 40-dB difference in signal strength, a relative error of 10^{-3} can be achieved with six terms, and 10^{-6} can be achieved with nine terms. Fewer terms are required for smaller thresholds and pole spreads.

The following recursion generates $P_2^{(i)}(\delta_i)$ for $1 \leq i \leq N_R$, and $P_2^{(N_R)}(\delta_{N_R})$ is the desired $\det(\mathbf{H})$. Start it by defining the vectors

$$\mathbf{a}_{l_k}^{(i)} = \left[a_0 \delta_i^0, \ldots, a_{L_x^{(i)}-k-1} \delta_i^{L_x^{(i)}-k-1} \right]^{\mathrm{T}} \tag{14}$$

$$\mathbf{b}_{l_k}^{(i)} = \left[b_0^{(i)} \delta_i^0, \ldots, b_{L_x^{(i)}-k-1}^{(i)} \delta_i^{L_x^{(i)}-k-1} \right]^{\mathrm{T}} \tag{15}$$

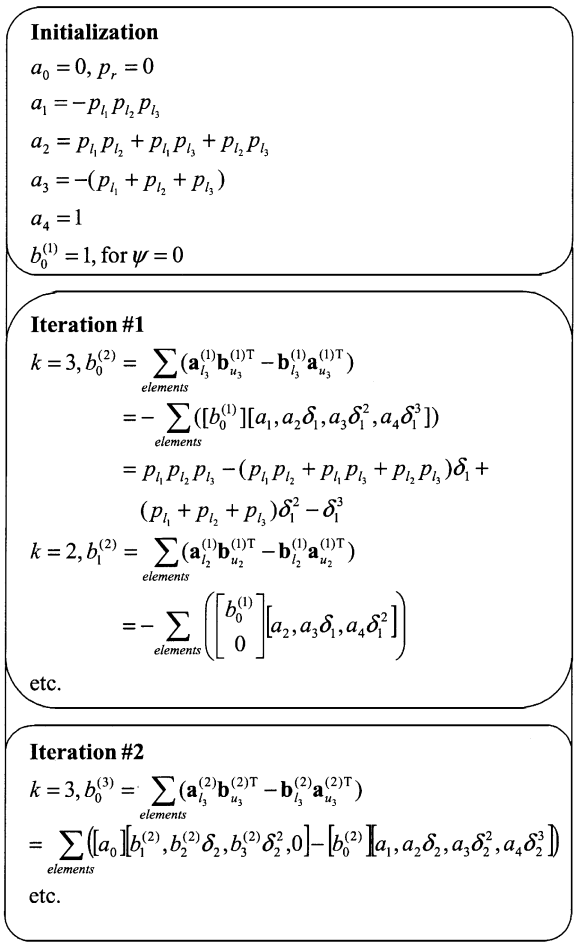

Fig. 2. Algorithm implementation for three pole pairs.

$$\mathbf{a}_{u_k}^{(i)} = \left[a_{L_x^{(i)}-k} \delta_i^0, \ldots, a_{L_x^{(i)}} \delta_i^k \right]^{\mathrm{T}} \tag{16}$$

$$\mathbf{b}_{u_k}^{(i)} = \left[b_{L_x^{(i)}-k}^{(i)} \delta_i^0, \ldots, b_{L_x^{(i)}}^{(i)} \delta_i^k \right]^{\mathrm{T}} \tag{17}$$

where $L_x^{(i)}$ is the length of $P_1(\delta_i)$ or $P_2^{(i)}(\delta_i)$, whichever is larger. The range of k is $0 \leq k \leq L_x^{(i)} - 1$, and the subscripts, l and u denote lower and upper, respectively. The coefficients of the next polynomial $P_2^{(i+1)}(\delta_{i+1})$ are calculated as

$$b_{L_x^{(i)}-k-1}^{(i+1)} = \sum_{\text{elements}} \left(\mathbf{a}_{l_k}^{(i)} \mathbf{b}_{u_k}^{(i)\mathrm{T}} - \mathbf{b}_{l_k}^{(i)} \mathbf{a}_{u_k}^{(i)\mathrm{T}} \right). \tag{18}$$

The summation in (18) denotes a summation over all of the elements in the resulting matrix. An example of the implementation for three pole pairs is given in Fig. 2. For simplicity, the parameter p_r is set to zero. As can be seen from the figure, the coefficients of $P_2^{(2)}(\delta_2)$ are easily calculated from $P_1(\delta_1)$ and $P_2^{(1)}(\delta_1)$, and the coefficients of $P_2^{(3)}(\delta_3)$ are calculated from $P_1(\delta_2)$ and $P_2^{(2)}(\delta_2)$. In the algorithm's final iteration, $i = N_R$, and $b_w^{(N_R)}$ are the coefficients of $P_2^{(N_R)}(\delta_{N_R})$. The CDF of the random variable, f, is then given by $P(f < \psi) = -(A/G)(e^{p_r \psi}) P_2^{(N_R)}(\delta_{N_R})$, and numerical values are used for δ_{N_R}. The entire procedure requires $N_R - 1$ iterations to invert a characteristic function with N_R RP poles.

For clarity, (11) shows $\det(\mathbf{H})$ multiplied by $-(A/G)e^{p_r \psi}$ once the iterations are complete. However, (3) shows that division by G can be performed one stage with each iteration, a stabilizing feature when pole multiplicities are very high.

IV. Conclusion

To conclude, (18) provides an efficient and numerically stable way of calculating error and erasure rates when the detection scheme can be expressed as a quadratic form in zero-mean complex Gaussian random variables. The method is very general, since it can invert a characteristic function with simple, multiple, and nearly equal poles. The characteristic function is inverted without resorting to numerical integration or tedious differentiation by hand.

References

[1] L. R. Welburn and J. K. Cavers, "Pilot signals improve the performance of a Reed–Solomon errors and erasures decoder in Rayleigh fading channels," *IEEE Trans. Commun.*, vol. 47, pp. 689–696, May 1999.

[2] M. Schwartz, W. Bennett, and S. Stein, *Communication Systems and Techniques*. New York: McGraw-Hill, 1966.

[3] J. K. Cavers and P. Ho, "Analysis of the error performance of trellis-coded modulations in Rayleigh-fading channels," *IEEE Trans. Commun.*, vol. 40, pp. 74–83, Jan. 1992.

[4] K. S. Hwang and F. C. Chang, "A numerical method for solving time-invariant system by the generalized inverse Vandermonde matrix," *J. Inform. Sci. Eng.*, vol. 15, pp. 321–327, 1999.

[5] E. Kreyszig, *Advanced Engineering Mathematics*, 6th ed. New York: Wiley, 1988.

[6] L. Welburn, "Macrodiversity and multiuser detection: High performance tools in a DS-CDMA system," Ph.D. dissertation, Simon Fraser Univ., Burnaby, BC, Canada, May 2001.