Technical Note: Eigenvalue Derivation

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In this note, the eigenvalues of the matrix product $2\mathbf{RQ}$ defined in [1] are derived. The $(L+1)\times(L+1)$ matrices $\mathbf{R}$ and $\mathbf{Q}$ are given by

$$\mathbf{R} = N_o \begin{bmatrix} 1+\Gamma & \Gamma/(AL) & \Gamma/(AL) & \cdots & \Gamma/(AL) \\ \Gamma/(AL) & \Gamma/\left(A^2L\right) & 0 & \cdots & 0 \\ \Gamma/(AL) & 0 & \Gamma/\left(A^2L\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma/(AL) & 0 & 0 & \cdots & \Gamma/\left(A^2L\right) \end{bmatrix} \tag{1}$$

and $\mathbf{Q} = \mathbf{F} + \mathbf{F}^\dagger$ where

$$\mathbf{F} = \begin{bmatrix} 0 & A(1-c_1) & A(1-c_2) & \cdots & A(1-c_L) \\ 0 & 0 & A^2(c_1^*c_2-1) & \cdots & A^2(c_1^*c_L-1) \\ 0 & 0 & 0 & \cdots & A^2(c_2^*c_L-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{2}$$

Γ is the signal-to-noise ratio, N_o is the noise variance, A is the amplitude of the signal on each transmit antenna, L is the number of transmit antennas, and c_l is the PSK symbol transmitted on the lth antenna with the property that

$$|c_l|^2 = (\mathrm{Re}[c_l])^2 + (\mathrm{Im}[c_l])^2 = 1. \tag{3}$$

As stated in [1], the rank of the product $2\mathbf{RQ}$ is only two, thus it has only two nonzero eigenvalues. Consequently, the characteristic polynomial may be expressed as

$$\det[2\mathbf{RQ} - \lambda\mathbf{I}] = \lambda^{L-1}\left(\lambda^2 + \beta_1\lambda + \beta_2\right) = 0. \tag{4}$$

According to Bôcher's formula [2], the coefficients of the quadratic polynomial are given by $\beta_1 = -T_1$ and $\beta_2 = -\frac{1}{2}(\beta_1 T_1 + T_2)$ where $T_n = trace\left[(2\mathbf{RQ})^n\right]$. Knowing the coefficients allows one to calculate the eigenvalues as

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = \frac{1}{2}\left[T_1 \pm \sqrt{2T_2 - T_1^2}\right]. \tag{5}$$

Evaluation of the two matrix traces T_1 and T_2 — and thus the eigenvalues — is made easier if the matrix $\mathbf{R}$ is written as $\mathbf{R} = N_o(\mathbf{R}_1 + \mathbf{R}_2)$ where

$$\mathbf{R}_1 = \begin{bmatrix} 1+\Gamma & \Gamma/(AL) & \Gamma/(AL) & \cdots & \Gamma/(AL) \\ \Gamma/(AL) & 0 & 0 & \cdots & 0 \\ \Gamma/(AL) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Gamma/(AL) & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{6}$$

and

$$\mathbf{R}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Gamma/\left(A^2L\right) & 0 & \cdots & 0 \\ 0 & 0 & \Gamma/\left(A^2L\right) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Gamma/\left(A^2L\right) \end{bmatrix}. \tag{7}$$

Thus T_1 and T_2 are given by

$$T_1 = 2N_o trace\left[\mathbf{R}_1\mathbf{F} + \mathbf{R}_1\mathbf{F}^\dagger + \mathbf{R}_2\mathbf{F} + \mathbf{R}_2\mathbf{F}^\dagger\right] \tag{8}$$

and

$$T_2 = 4N_o^2 trace\left[\left(\mathbf{R}_1\mathbf{F} + \mathbf{R}_1\mathbf{F}^\dagger + \mathbf{R}_2\mathbf{F} + \mathbf{R}_2\mathbf{F}^\dagger\right)\left(\mathbf{R}_1\mathbf{F} + \mathbf{R}_1\mathbf{F}^\dagger + \mathbf{R}_2\mathbf{F} + \mathbf{R}_2\mathbf{F}^\dagger\right)\right]. \tag{9}$$

The above expressions require evaluation of the traces of several matrix products: four twofold products in the former, and 16 fourfold products in the latter. Fortunately, the spare nature and special form of the matrices make some of the traces zero, and limit the complexity of others.

In the derivation of the matrix traces, it will become important to focus on the following four twofold products which are shown for the case of $L = 3$:

$$\mathbf{R}_1\mathbf{F} = \begin{bmatrix} 0 & A(1+\Gamma)(1-c_1) & \begin{matrix} A(1+\Gamma)(1-c_2) \\ +A(\Gamma/L)(c_1^*c_2-1) \end{matrix} & \begin{matrix} A(1+\Gamma)(1-c_3) \\ +A(\Gamma/L)(c_1^*c_3-1) + A(\Gamma/L)(c_2^*c_3-1) \end{matrix} \\ 0 & (\Gamma/L)(1-c_1) & (\Gamma/L)(1-c_2) & (\Gamma/L)(1-c_3) \\ 0 & (\Gamma/L)(1-c_1) & (\Gamma/L)(1-c_2) & (\Gamma/L)(1-c_3) \\ 0 & (\Gamma/L)(1-c_1) & (\Gamma/L)(1-c_2) & (\Gamma/L)(1-c_3) \end{bmatrix}, \tag{10}$$

$$\mathbf{R}_1\mathbf{F}^\dagger = \begin{bmatrix} (\Gamma/L)\left[(1-c_1)+(1-c_2)+(1-c_3)\right] & A(\Gamma/L)\left[(c_1c_2^*-1)+(c_1c_3^*-1)\right] & A(\Gamma/L)(c_2c_3^*-1) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{11}$$

$$\mathbf{R}_2\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\Gamma/L)(c_1^*c_2-1) & (\Gamma/L)(c_1^*c_3-1) \\ 0 & 0 & 0 & (\Gamma/L)(c_2^*c_3-1) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{12}$$

and

$$\mathbf{R}_2\mathbf{F}^\dagger = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (\Gamma/(AL))(1-c_1^*) & 0 & 0 & 0 \\ (\Gamma/(AL))(1-c_2^*) & (\Gamma/(AL))(c_1c_2^*-1) & 0 & 0 \\ (\Gamma/(AL))(1-c_3^*) & (\Gamma/(AL))(c_1c_3^*-1) & (\Gamma/(AL))(c_2c_3^*-1) & 0 \end{bmatrix}. \tag{13}$$

The generalization to arbitrary L will be stated later.

The following identities will be useful in the derivation of the traces

$$trace\,[AB] = trace\,[BA] \tag{14a}$$

$$trace\left[A^\dagger\right] = (trace\,[A])^* \tag{14b}$$

$$\left(\sum_{n=1}^{N} x_n\right)^2 = \sum_{n=1}^{N}\sum_{m=1}^{N} x_n x_m = \sum_{n=1}^{N} x_n^2 + 2\sum_{n=1}^{N-1}\sum_{m=n+1}^{N} x_n x_m. \tag{15}$$

1 Derivation of T_1

Observing (12), one can see that the diagonal elements are all zero and thus $trace\left[\mathbf{R}_2\mathbf{F}\right] = 0$. Using this fact, and property (14b), (8) may be written as

$$\begin{aligned} T_1 &= 2N_o\left(trace\left[\mathbf{R}_1\mathbf{F}\right] + \left(trace\left[\mathbf{R}_1\mathbf{F}\right]\right)^*\right) \\ &= 4N_o \operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}\right]\right\}. \end{aligned} \tag{16}$$

It is easy to show that for arbitrary L, the elements of the main diagonal of the matrix product $\mathbf{R}_1\mathbf{F}$ are all of the form $(\Gamma/L)(1-c_l)$, thus T_1 is given simply by

$$T_1 = 4N_o\left(\frac{\Gamma}{L}\right)\sum_{l=1}^{L}\left(1 - \operatorname{Re}\left[c_l\right]\right). \tag{17}$$

2 Derivation of T_2

Observing (9), one can see that T_2 involves 16 terms; however, we can simplify the expression somewhat by using properties (14a) and (14b) to combine the cross terms resulting in

$$\begin{aligned} T_2 = {} & 4N_o^2\left(trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_1\mathbf{F}\right] + trace\left[\mathbf{R}_1\mathbf{F}^\dagger\mathbf{R}_1\mathbf{F}^\dagger\right] + trace\left[\mathbf{R}_2\mathbf{F}\mathbf{R}_2\mathbf{F}\right] + trace\left[\mathbf{R}_2\mathbf{F}^\dagger\mathbf{R}_2\mathbf{F}^\dagger\right]\right. \\ & +2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_1\mathbf{F}^\dagger\right]\right\} + 2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}\right]\right\} + 2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right]\right\} \\ & +2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}^\dagger\mathbf{R}_2\mathbf{F}\right]\right\} + 2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}^\dagger\mathbf{R}_2\mathbf{F}^\dagger\right]\right\} \\ & \left.+2\operatorname{Re}\left\{trace\left[\mathbf{R}_2\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right]\right\}\right). \end{aligned} \tag{18}$$

Close observation of (10)–(13) reveals that the traces of the fourfold products in the 3rd 4th, 5th, and 8th terms in the above expression are zero. Discarding these terms and combining the first and second terms as well as the 6th and 9th terms, using properties (14a) and (14b), results in the following simplified expression for T_2

$$T_2 = 8N_o^2\operatorname{Re}\left\{trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_1\mathbf{F}\right] + 2trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}\right] + trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right] + trace\left[\mathbf{R}_2\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right]\right\}. \tag{19}$$

Indeed, the special structure of the matrices has resulted in a large amount of simplification: a reduction in the number of terms from 16 to four.

Evaluation of the four terms may proceed by using (10)–(13) to calculate the elements on the main diagonal of each fourfold product in (19). Although the result applies to the case of $L = 3$, it is a simple matter to generalize to arbitrary L. For the case of $L = 3$, the first term is given by

$$\begin{aligned} trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_1\mathbf{F}\right] = {} & \left(\frac{\Gamma}{L}\right)^2\left\{(1-c_1)\left[(1-c_1)+(1-c_2)+(1-c_3)\right]\right. \\ & +(1-c_2)\left[(1-c_1)+(1-c_2)+(1-c_3)\right] \\ & \left.+(1-c_3)\left[(1-c_1)+(1-c_2)+(1-c_3)\right]\right\}. \end{aligned} \tag{20}$$

The second term is given by

$$\begin{aligned} 2trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}\right] = {} & 2\left(\frac{\Gamma}{L}\right)^2\left[(1-c_1)\left(c_1^*c_2-1\right)+(1-c_1)\left(c_1^*c_3-1\right)\right. \\ & \left.+(1-c_2)\left(c_2^*c_3-1\right)\right]. \end{aligned} \tag{21}$$

The third term is given by

$$\begin{aligned} trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right] = {} & \left(\frac{\Gamma}{L}\right)(1+\Gamma)\left[|1-c_1|^2+|1-c_2|^2+|1-c_3|^2\right] \\ & +2\left(\frac{\Gamma}{L}\right)^2\operatorname{Re}\left[(1-c_2)\left(c_1c_2^*-1\right)+(1-c_3)\left(c_1c_3^*-1\right)+(1-c_3)\left(c_2c_3^*-1\right)\right], \end{aligned} \tag{22}$$

and the fourth term is given by

$$trace\left[\mathbf{R}_2\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right] = \left(\frac{\Gamma}{L}\right)^2 \left[\left|c_1^* c_2 - 1\right|^2 + \left|c_1^* c_3 - 1\right|^2 + \left|c_2^* c_3 - 1\right|^2\right]. \tag{23}$$

Observing (20)–(23) one can see that the four terms generalize as follows

$$\begin{aligned} trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_1\mathbf{F}\right] &= \left(\frac{\Gamma}{L}\right)^2 \sum_{l=1}^{L}\sum_{m=1}^{L} (1-c_l)(1-c_m) \\ &= \left(\frac{\Gamma}{L}\right)^2 \left[\sum_{l=1}^{L}(1-c_l)^2 + 2\sum_{l=1}^{L-1}\sum_{m=l+1}^{L}(1-c_l)(1-c_m)\right], \end{aligned} \tag{24}$$

$$2trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}\right] = 2\left(\frac{\Gamma}{L}\right)^2 \sum_{l=1}^{L-1}\sum_{m=l+1}^{L}(1-c_l)\left(c_l^* c_m - 1\right), \tag{25}$$

$$\begin{aligned} trace\left[\mathbf{R}_1\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right] &= \left(\frac{\Gamma}{L}\right)(1+\Gamma)\sum_{l=1}^{L}|1-c_l|^2 + 2\left(\frac{\Gamma}{L}\right)^2 \sum_{l=1}^{L-1}\sum_{m=l+1}^{L} \mathrm{Re}\left[(1-c_m)\left(c_l c_m^* - 1\right)\right] \\ &= \left(\frac{\Gamma}{L}\right)\sum_{l=1}^{L}|1-c_l|^2 + \left(\frac{\Gamma}{L}\right)^2\left[L\sum_{l=1}^{L}|1-c_l|^2 \right. \\ &\left. +2\sum_{l=1}^{L-1}\sum_{m=l+1}^{L}\mathrm{Re}\left[(1-c_m)\left(c_l c_m^* - 1\right)\right]\right] \end{aligned} \tag{26}$$

and

$$trace\left[\mathbf{R}_2\mathbf{F}\mathbf{R}_2\mathbf{F}^\dagger\right] = \left(\frac{\Gamma}{L}\right)^2 \sum_{l=1}^{L-1}\sum_{m=l+1}^{L}\left|c_l^* c_m - 1\right|^2. \tag{27}$$

Further expansion of the above general expressions and substitution of the results into (19) yields the following expression for T_2

$$\begin{aligned} T_2 &= 8N_o^2\left[2\left(\frac{\Gamma}{L}\right)\sum_{l=1}^{L}(1-\mathrm{Re}[c_l]) \right. \\ &+ \left(\frac{\Gamma}{L}\right)^2\left(\sum_{l=1}^{L}\left(2L(1-\mathrm{Re}[c_l]) + (1-\mathrm{Re}[c_l])^2 + (\mathrm{Re}[c_l])^2 - 1\right) \right. \\ &\left.\left. +2\sum_{l=1}^{L-1}\sum_{m=l+1}^{L}\left((1-\mathrm{Re}[c_l])(1-\mathrm{Re}[c_m]) + \mathrm{Re}[c_l]\,\mathrm{Re}[c_m] - 1\right)\right)\right]. \end{aligned} \tag{28}$$

In the derivation of this expression, explicit use is made of property (3) for PSK signals.

3 Derivation of Eigenvalues

Equation (5) for the eigenvalues requires evaluation of the term $T_2 - T_1^2$ which, using (17), is given by

$$\begin{aligned} 2T_2 - T_1^2 &= 2T_2 - 16N_o^2\left(\frac{\Gamma}{L}\right)^2\left(\sum_{l=1}^{L}(1-\mathrm{Re}[c_l])\right)^2 \\ &= 2T_2 - 16N_o^2\left(\frac{\Gamma}{L}\right)^2\left[\sum_{l=1}^{L}(1-\mathrm{Re}[c_l])^2 + 2\sum_{l=1}^{L-1}\sum_{m=l+1}^{L}(1-\mathrm{Re}[c_l])(1-\mathrm{Re}[c_m])\right]. \end{aligned} \tag{29}$$

Substituting equation (28) for T_2 into this expression results in

$$\begin{aligned} 2T_2 - T_1^2 &= 16N_o^2 \left[2\left(\frac{\Gamma}{L}\right) \sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]) \right. \\ &\left. + \left(\frac{\Gamma}{L}\right)^2 \left(\sum_{l=1}^{L} \left(2L(1 - \mathrm{Re}[c_l]) + (\mathrm{Re}[c_l])^2 - 1 \right) + 2 \sum_{l=1}^{L-1} \sum_{m=l+1}^{L} (\mathrm{Re}[c_l]\,\mathrm{Re}[c_m] - 1) \right) \right] \end{aligned} \quad (30)$$

Expanding gives

$$\begin{aligned} 2T_2 - T_1^2 &= 16N_o^2 \left[2\left(\frac{\Gamma}{L}\right) \sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]) \right. \\ & + \left(\frac{\Gamma}{L}\right)^2 \left(2L^2 - 2L \sum_{l=1}^{L} \mathrm{Re}[c_l] + \sum_{l=1}^{L} (-1) + 2 \sum_{l=1}^{L-1} \sum_{m=l+1}^{L} (-1) \right. \\ & \left. \left. + \sum_{l=1}^{L} (\mathrm{Re}[c_l])^2 + 2 \sum_{l=1}^{L-1} \sum_{m=l+1}^{L} \mathrm{Re}[c_l]\,\mathrm{Re}[c_m] \right) \right]. \end{aligned} \quad (31)$$

Use of the identity (15) allows this to be rewritten as

$$\begin{aligned} 2T_2 - T_1^2 &= 16N_o^2 \left[2\left(\frac{\Gamma}{L}\right) \sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]) \right. \\ & \left. + \left(\frac{\Gamma}{L}\right)^2 \left(L^2 - 2L \sum_{l=1}^{L} \mathrm{Re}[c_l] + \sum_{l=1}^{L} \sum_{m=1}^{L} \mathrm{Re}[c_l]\,\mathrm{Re}[c_m] \right) \right] \\ &= 16N_o^2 \left[2\left(\frac{\Gamma}{L}\right) \sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]) + \left(\frac{\Gamma}{L}\right)^2 \left(\sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]) \right)^2 \right] \end{aligned} \quad (32)$$

Finally, substituting this result along with (17) into (5) gives the eigenvalues as

$$\begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix} = 2N_o \left[a\left(\frac{\Gamma}{L}\right) \pm \sqrt{a^2 \left(\frac{\Gamma}{L}\right)^2 + 2a\left(\frac{\Gamma}{L}\right)} \right] \quad (33)$$

where a is defined as

$$a = \sum_{l=1}^{L} (1 - \mathrm{Re}[c_l]). \quad (34)$$

References

[1] S.J. Grant and J.K. Cavers, "A method for increasing downlink capacity by coded multiuser transmission with a base station antenna array," submitted to *IEEE Transactions on Communications*, July 2000.

[2] P. DeRusso, R. Roy, and C. Close, *State Variables for Engineers*, New York: John Wiley and Sons, 1965.