ENSC327
Communications Systems
19: Random Processes

Jie Liang
School of Engineering Science
Simon Fraser University

Outline

- Random processes
- Stationary random processes
- Autocorrelation of random processes

Definition of Random Process

- A deterministic process has only one possible 'reality' of how the process evolves under time.
- In a stochastic or random process there are some uncertainties in its future evolution described by probability distributions.
- Even if the initial condition (or starting point) is known, there are many possibilities the process might go to, but some paths are more probable and others less.
- http://en.wikipedia.org/wiki/Stochastic_process

Definition of Random Process

- Many time-varying signals are random in nature:
  - Noises
  - Image, audio: usually unknown to the distant receiver.

- Random process represents the mathematical model of these random signals.
- Definition: A random process (or stochastic process) is a collection of random variables (functions) indexed by time.
- Notation:

$$X(t,s)$$

  - s: the sample point of the random experiment.
  - t: time.
- Simplified notation:

$$X(t)$$

Random Processes

- The difference between random variable and random process:
  - Random variable: an outcome is mapped to a number.
  - Random process: an outcome is mapped to a random waveform that is a function of time

- We are interested in the ways that these time functions evolve
  - correlation
  - spectra
  - linear systems

Cont…

- For a fixed sample point s_j, $X(t, s_j)$ is a realization or sample function of the random process.
  - Simplified Notation:

$$X(t, s_j) \text{ is denoted as } x_j(t).$$

- For a fixed time t_k, the set of numbers

$$\{X(t_k, s_1), \ldots, X(t_k, s_n)\} = \{x_1(t_k), \ldots, x_n(t_k)\}$$

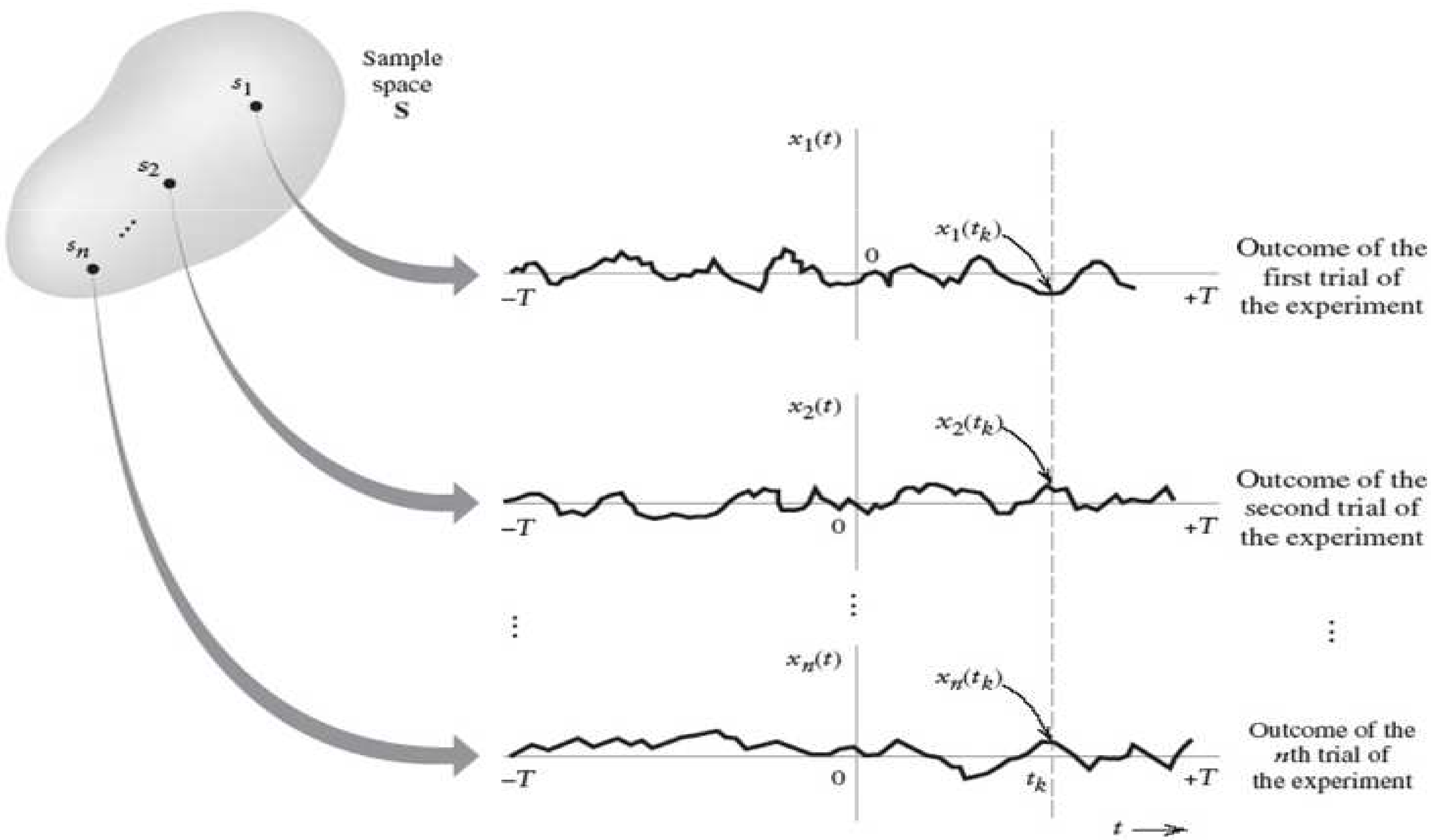
is a random variable, denoted by $X(t_k)$.

- For a fixed s_j and t_k, $X(t_k, s_j)$ is a number.

Pictorial View

Each sample point represents a time-varying function.

Ensemble: The set of all time-varying functions.



Examples of Random Processes

1. $X(t,s) = Y(s)f(t)$ or $X(t,s) = Yf(t)$ for short.
   - Y: a random variable.
   - f: a deterministic function of parameter t.
2. $X(t,s) = A(s)\cos(2\pi f_0 t + Q)$ or $X(t) = A\cos(2\pi f_0 t + Q)$.
   - A: a random variable.
   - Q : a random variable.
3. $X(t) = \sum_n X(n) p_n(t - T(n))$
   - X(n), T(n): random sequences.
   - $p_n(t)$: deterministic waveforms.

Probability Distribution of a Random Process

- For any random process, its probability distribution function is uniquely determined by its finite dimensional distributions.
- The k dimensional cumulative distribution function of a process is defined by

$$F_{X(t_1),\dots,X(t_k)}\left(x_1,\dots,x_k\right)=P\left(X(t_1)\le x_1,\dots,X(t_k)\le x_k\right)$$

  for any $t_1,\dots,t_k$ and any real numbers $x_1, \dots, x_k$.

- The cumulative distribution function tells us everything we need to know about the process $\{X_t\}$.

Outline

- Random processes
- Stationary random processes
- Autocorrelation of random processes

Stationarity

- In general, the time-dependent N-fold joint pdf's are needed to describe a random process for all possible N:
  - Very difficult to obtain all pdf's.
- The analysis can be simplified if the statistics are time independent.
- The random process is called first-order stationary if

$F_{X(t)}(x)$: the CDF of the random process X(t) at a fixed time t.

Stationarity

the pdf $f_{X(t)}(x)$ is independent of time $\Rightarrow$

- The mean and variance of the first-order stationary random process are independent of time:

$$\mu_{X(t_1)} = \mu_{X(t_1+\tau)}, \qquad \sigma^2_{X(t_1)} = \sigma^2_{X(t_1+\tau)}.$$

- Proof:

Stationarity

- The random process is called second-order stationary if the 2nd order CDF is independent of time:

Strict Stationarity

- A strictly stationary process (or strongly stationary process, or stationary process) is a stochastic process whose joint pdf does not change when shifted in time.
- Definition: a random process X(t) is said to be stationary if, for all k, for all τ, and for all $t_1, t_2, \ldots, t_k$,

Strict Stationarity

- An example of strictly stationary process is one in which all $X(t_i)$'s are mutually Independent and Identically Distributed.
- Such a random process is called IID random process.
- In this case,

- Since the joint pdf above does not depend on the times {ti}, the process is strictly stationary.
- An example of IID process is white noise (studied later)
  - Widely used in communications theory

Outline

- Random processes
- Stationary random processes
- Autocorrelation of random processes

Correlation of Random Processes

- Recall: Covariance of two random variables:

$$Cov(X,Y) = E\{[X-\mu_X][Y-\mu_Y]\} = E\{XY\} - \mu_X \mu_Y$$

Consider $X(t_1)$ and $X(t_2)$: samples of X(t) at t_1 and t_2.

$X(t_1)$ and $X(t_2)$ are both random variables

So we can also define their covariance:

$$Cov(X(t_1), X(t_2)) = E\{X(t_1)X(t_2)\} - \mu_{X(t_1)}\mu_{X(t_2)}$$

Correlation of Random Processes

- Recall: autocorrelation of deterministic energy signals:

$$R_x(\tau) \stackrel{\Delta}{=} \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt$$

- Similarly, the autocorrelation of deterministic power signal is:

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x^*(t-\tau)dt$$

- The limit is necessary since the energy of the power signal can be infinite.

Correlation of Random Processes

- For random processes: need to consider probability distributions.
- The autocorrelation function of a random process:

- Note: steps to get $E\{X(t_1)X^*(t_2)\}$:
- 1: For each sample function $X(t, s_j)$, calculate $X(t_1, s_j)\ X^*(t_2, s_j)$.
- 2: Take weighted average over all possible sample functions s_j.
- (See Example 1 later)
- If X(t) is stationary to the 2nd order or higher order, $R_X(t1,t2)$ only depends on the time difference $t_1 - t_2$, so it can be written as a single variable function:

Wide-Sense Stationarity (WSS)

- In many cases we do not require a random process to have all of the properties of the 2nd order stationarity.

- A random process is said to be **wide-sense stationary** or weakly stationary if and only if

$$R_X(t,s) = E\{X(t)X^*(s)\} = R_X(t-s).$$

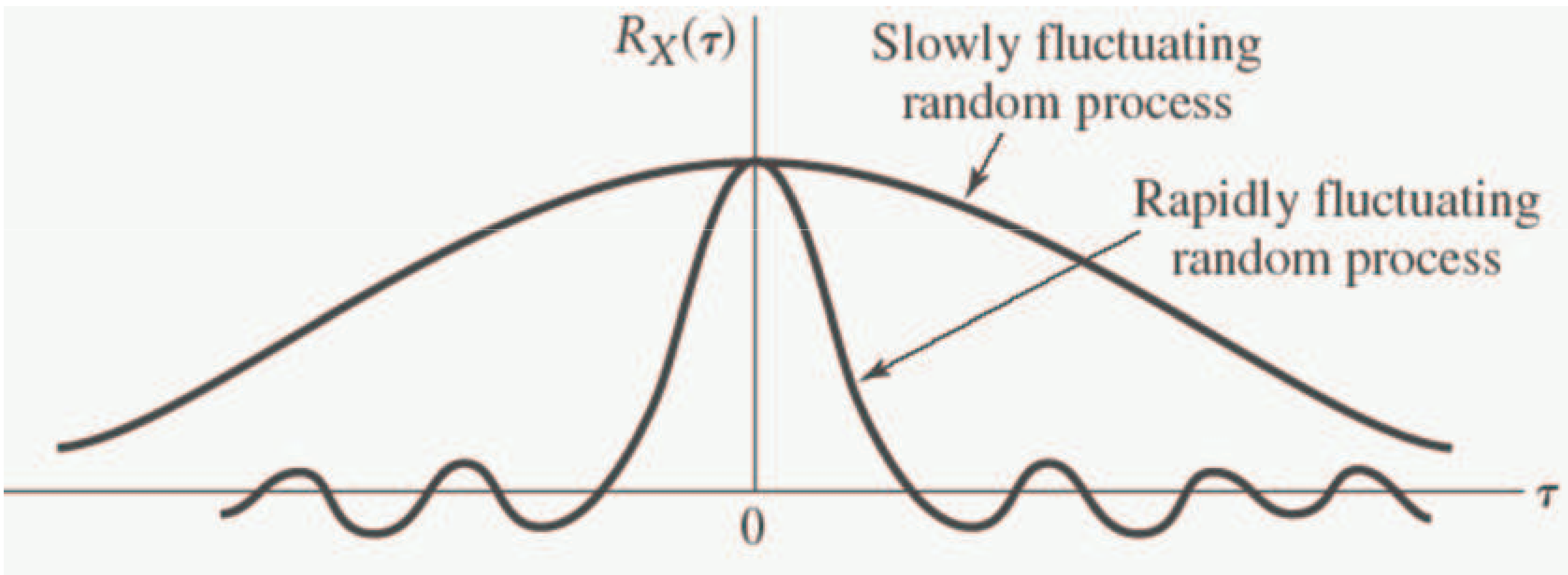
Property of Autocorrelation

- For real-valued wide-sense stationary X(t), we have:
- 1. $$R_X(0) = E\{X^2(t)\}.$$
- 2. $R_X(\tau)$ is even symmetry: $R_X(-\tau) = R_X(\tau)$.
- Proof:
- 3. $R_X(\tau)$ is max at the origin $\tau = 0$.
- Proof:

$$R_X(t,s)=E\left\{X(t)X^*(s)\right\}=R_X(t-s).$$

Property of Autocorrelation

- Example of the autocorrelation function:



- Symmetric, peak at 0.
- Change slow if X(t) changes slow.

Example 1

$$X(t) = A\cos(2\pi ft + \Theta)$$

A: constant. Θ: uniform random variable in $[0, 2\pi]$.

- Find the autocorrelation of X.
- Is X wide-sense stationary?

Solution:

Example 1

Y = g(X) ➔ $\mu_Y = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

Example 2

$$X(t) = A\cos(2\pi ft)$$

A: uniform random variable in [0, 1].

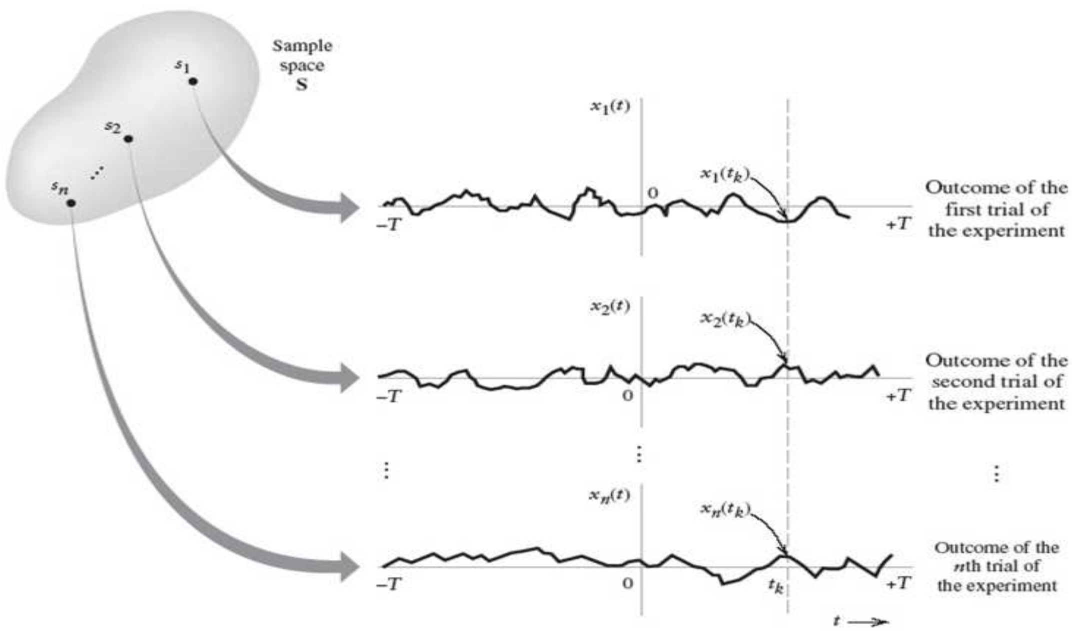
- Find the autocorrelation of X. Is X WSS?

Solution:

Example 2

$$X(t) = A\cos(2\pi f t)$$

Note the value of A at time t1 and t2 are same in this example, because it's determined by the random experiment.



Example 3

- A random process X(t) consists of three possible sample functions: $x_1(t)=1$, $x_2(t)=3$, and $x_3(t)=\sin(t)$. Each occurs with equal probability. Find its mean and auto-correlation. Is it wide-sense stationary?
- **Solution:**