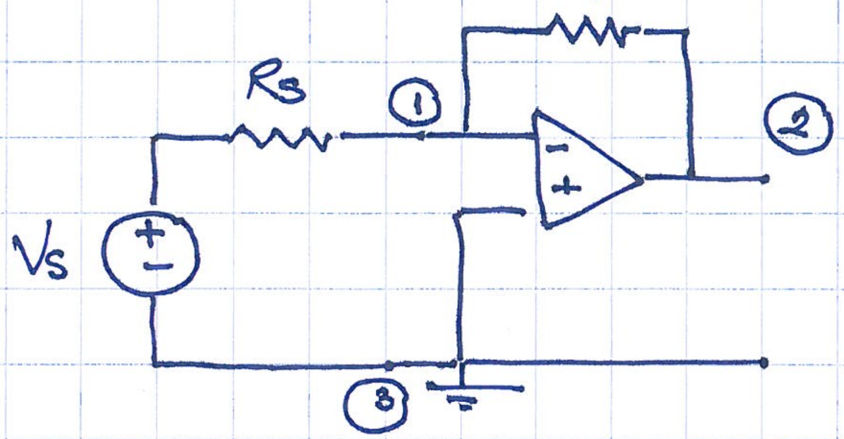
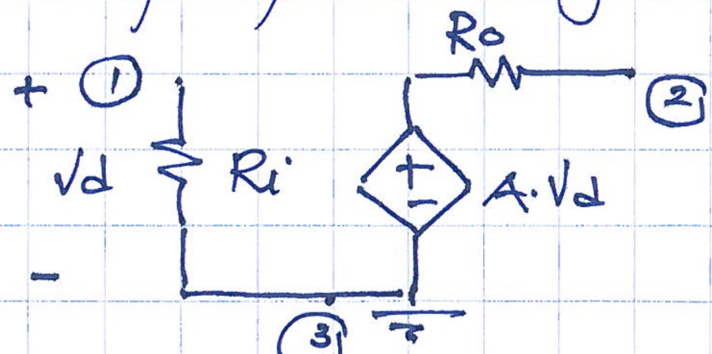
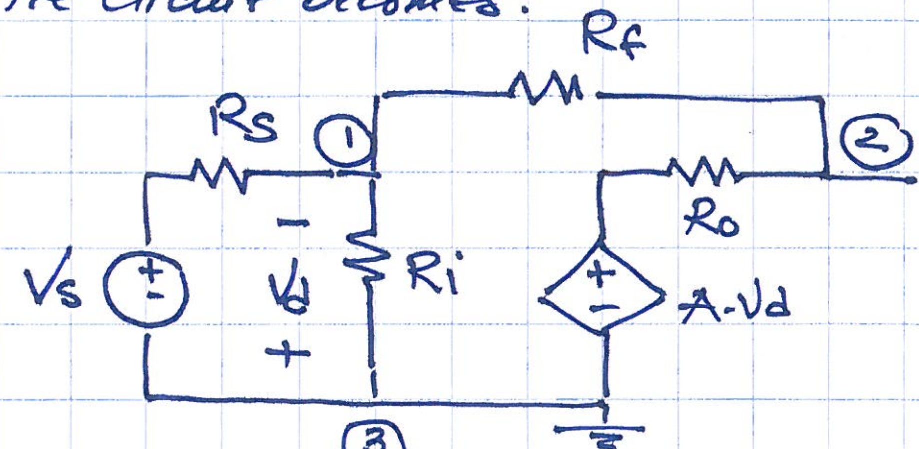
Problem #1:



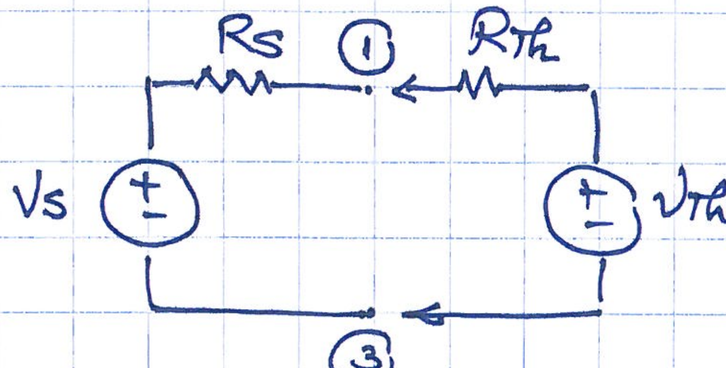
Replace the op-amp with the given model:



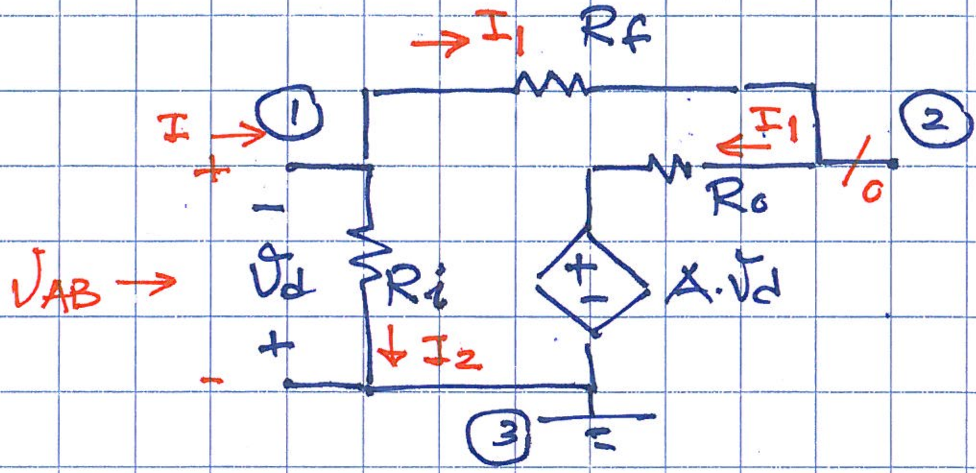
the circuit becomes:



Thevenin equivalent:

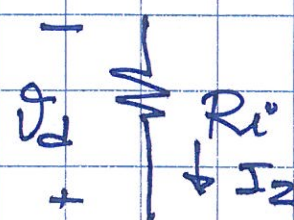


Thévenin equivalent Looking left to right between nodes 1 and 3:



$R_{Th} = ?$

$V_{Th} = ?$



KCL 1: $I - I_1 - I_2 = 0$

Ohm's Law: $I_2 = -v_d/R_i$

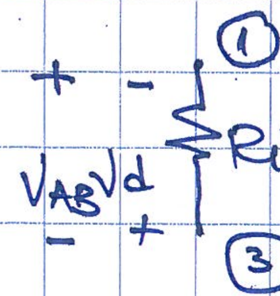
Hence,

$$I_1 - \frac{v_d}{R_i} = I \qquad (1)$$

KVL: $V_{AB} - R_f \cdot I_1 - R_o \cdot I_1 - A \cdot v_d = 0$

$$V_{AB} = (R_f + R_o) \cdot I_1 + A \cdot v_d$$

$V_{AB} = -v_d$ or $v_d = -V_{AB}$



Hence, $V_{AB} = (R_f + R_o) I_1 - A \cdot V_{AB}$

$$V_{AB}(1+A) = (R_f + R_o)\cdot I_1$$

$$V_{AB} = \frac{R_f + R_o}{1+A}\cdot I_1 \qquad (2)$$

$$R_{th} = \frac{V_{AB}}{I}$$

From (1): $I = I_1 - \frac{V_d}{R_i}$ Recall that $V_d = -V_{AB}$

$$I = I_1 + \frac{V_{AB}}{R_i}$$

$$I = I_1 + \frac{R_f + R_o}{1+A}\cdot I_1 \frac{1}{R_i}$$

$$I = \left(1 + \frac{R_f+R_o}{(1+A)R_i}\right)\cdot I_1 \qquad (3)$$

From (2) and (3):

$$R_{th} = \frac{V_{AB}}{I} \Rightarrow R_{th} = \frac{R_f+R_o}{1+A}\cdot I_1 \cdot \frac{1}{\left(1+\frac{R_f+R_o}{(1+A)R_i}\right)\cdot I_1}$$

$$R_{th} = \frac{R_f+R_o}{1+A}\cdot\frac{(1+A)\cdot R_i}{(1+A)\cdot R_i + R_f + R_o}$$

$$R_{th} = \frac{R_i(R_f+R_o)}{(1+A)\cdot R_i + R_f + R_o}$$

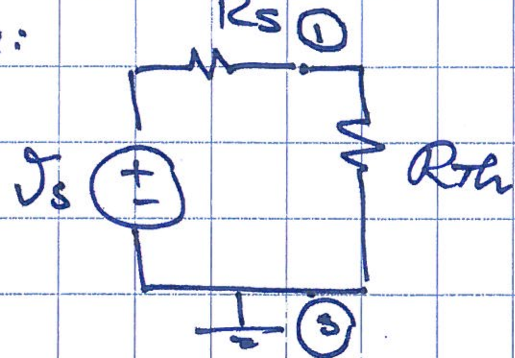
Simplif. $$R_{th} = \frac{1}{\frac{1+A}{R_f+R_o} + 1/R_i}$$

$v_{th} = ?$

Note that $v_{th} = 0$

voltage seen between 1 and 3 is ϕ (no sources are present)

Hence:



If the op-amp is ideal:

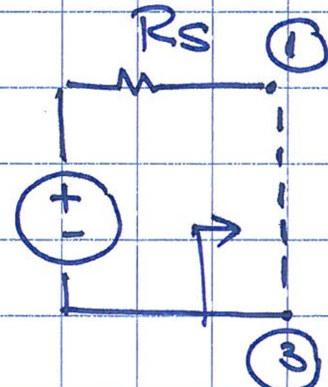
$R_i = \infty$

$R_o = 0$

$A = \infty$

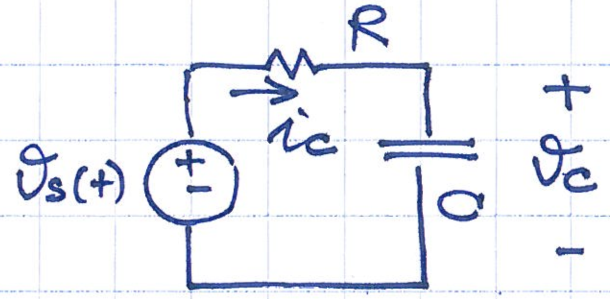
$$R_{th} = \frac{1}{\frac{1+\infty}{R_f+0} + \frac{1}{\infty}}$$

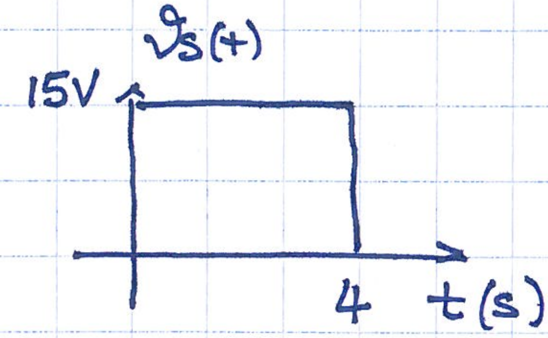
$R_{th} = 0$



virtual short circuit

Problem #2:





$v_s(t) = 15\left(u(t) - u(t-4)\right)$ V

$R = 200\ k\Omega$

$C = 10\,\mu F$

$v_C(0-) = 0$

State equations:

KVL: $v_s(t) - R \cdot i_c - v_c = 0$

$$i_c = C\frac{dv_c}{dt}$$

$$R \cdot C \cdot \frac{dv_c}{dt} + v_c = v_s$$

$$\frac{dv_c}{dt} + \frac{1}{RC} \cdot v_c = \frac{1}{RC} \cdot v_s$$

$RC = 200 \times 10^{3} \times 10 \times 10^{-6}$

$RC = 2$

$$\frac{dv_c}{dt} + \frac{1}{2} \cdot v_c = \frac{1}{2} \cdot v_s$$

State equations: $\frac{dv_c}{dt} + \frac{1}{2} \cdot v_c = \frac{15}{2}$ $\quad 0 < t < 4$

$$\frac{dv_c}{dt} + \frac{1}{2} \cdot v_c = 0 \quad t > 4$$

$0<t<4$:

$$\frac{dv_c}{dt}+\frac{1}{2}\cdot v_c=\frac{15}{2}$$

$$v_c=v_{ch}+v_{cp}$$

Homogeneous solution:

$$\frac{dv_{ch}}{dt}+\frac{1}{2}\cdot v_{ch}=0$$

$v_{ch}=ke^{st}$ $\quad$ k: constant

$\frac{dv_{ch}}{dt}=ks\cdot e^{st}$ $\quad$ s: constant

$k\cdot s\cdot e^{st}+\frac{1}{2}\cdot k\cdot e^{st}=0$; $k\neq 0$, $e^{st}\neq 0$

$s=-1/2$

$v_{ch}=ke^{-t/2}$

Particular solution:

$v_{cp}=A$ (right hand side is a constant)

$\frac{dv_{cp}}{dt}=0$

$0+\frac{1}{2}\cdot A=\frac{15}{2}$ $\quad$ $A=15$

Complete solution: $v_c=ke^{-t/2}+15$

$v_c(0-)=0$ (given)

$v_c(0+)=0$ $\quad$ Hence: $0=k+15$; $k=-15$

$$v_c=15(1-e^{-t/2})\quad 0<t<4$$

At $t=4s$:

$$V_c(4-) = 15\left(1-e^{-4/2}\right)$$

$$V_c(4^-) = 15\left(1-e^{-2}\right)$$

$$V_c(4^+) = 15\left(1-e^{-2}\right)$$

For $t>4s$

$$\frac{dV_c}{dt} + \frac{1}{2}\cdot V_c = 0$$

$$V_c = K_1 \cdot e^{-t/2}$$

the same as the homogeneous differential equation for $0<t<4$

Let $t' = t-4$

$$V_c = K_1 e^{-t'/2}$$

in time t':

$$V_c(0+) = K_1$$

$$V_c(0^+) = V_c(0^-)$$
$$= V_c(4^-)$$
$$= 15\left(1-e^{-2}\right)$$

$$K_1 = 15\left(1-e^{-2}\right)$$

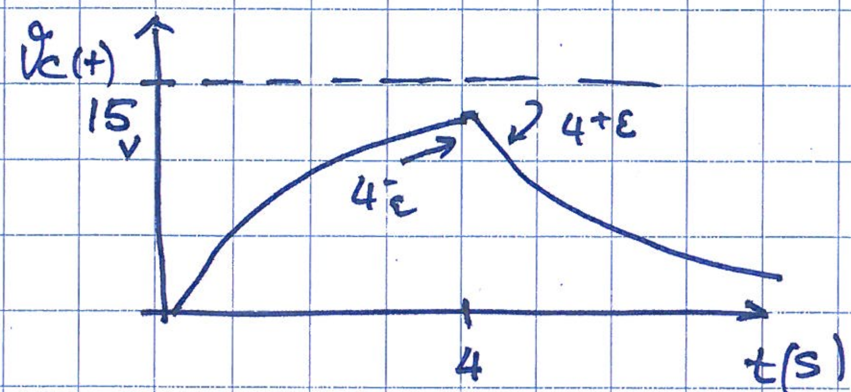
Hence:

$$V_c = 15\left(1-e^{-2}\right)e^{-t'/2}$$

$$V_c = 15\left(1-e^{-2}\right)e^{-\frac{t-4}{2}} \qquad t>4$$

$$V_c = 15\cdot\left(1-e^{-2}\right)\cdot e^{2}\cdot e^{-t/2}$$

$$V_c = 15\left(e^{2}-1\right)e^{-t/2} \qquad t>4$$



$$v_c(t) = \begin{cases} 0 & t<0 \\ 15(1-e^{-t/2}) & 0<t<4 \\ 15(e^2-1)e^{-t/2} & t>4 \end{cases}$$

Current:

$$i_c = C\frac{dv_c}{dt}$$

$$i_c = 10\times10^{-6}\cdot\frac{d}{dt}v_c$$

$$i_c = 0 \qquad t<0$$

$$i_c = 10\times10^{-6}\cdot\frac{d}{dt}\left(15\left(1-e^{-t/2}\right)\right) \qquad 0<t<4$$

$$i_c = 15\times10^{-5}\cdot(1/2)e^{-t/2}$$

$$i_c = 7.5\times10^{-5}e^{-t/2}$$

$$i_c = 75\times10^{-6}\cdot e^{-t/2} \qquad 0<t<4$$

$t > 4$

$$i_c = 10 \times 10^{-6} \cdot \frac{d}{dt} \left(15 (e^2 - 1) e^{-t/2} \right)$$

$$i_c = 10^{-5} \cdot 15 \cdot (e^2 - 1) \cdot (-1/2) \cdot e^{-t/2}$$

$$i_c = -75 (e^2 - 1) \cdot e^{-t/2} \times 10^{-6} \quad t > 4$$

Approximately: at $t = 4$

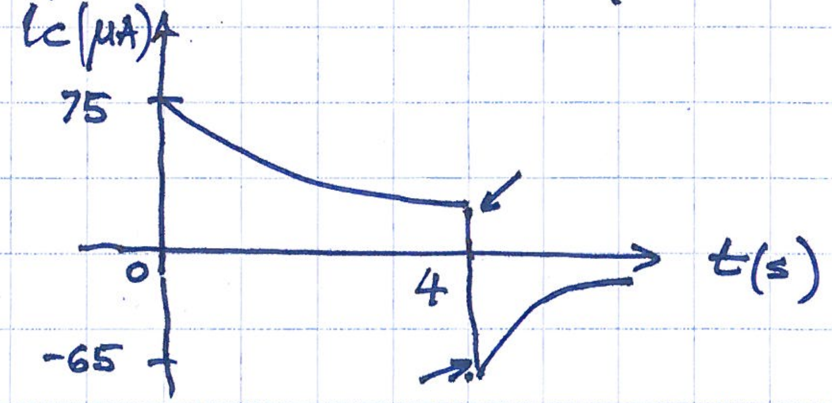
$$i_c(4^+) = -75 \cdot (e^2 - 1) \cdot e^{-2} \times 10^{-6}$$

$$= -75 \cdot (1 - e^{-2}) \times 10^{-6}$$

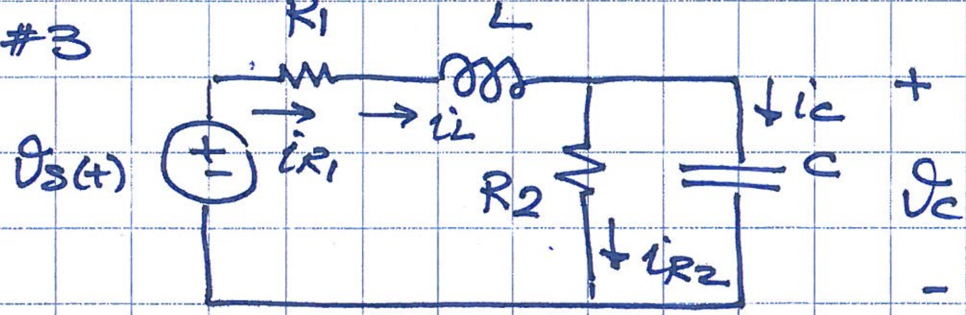
since $15(1 - e^{-2}) \approx 13$

$$i_c(4^+) = -75 \cdot 13/15 \times 10^{-6}$$

$$\approx -65 \, \mu A$$



Problem #3



$R_1 = 1\Omega$ $\quad v_s(t) = u(t)$

$R_2 = 1\Omega$

$L = \sqrt{2}\ H$

$C = \sqrt{2}\ F$

KCL: $i_{R_1} = i_L$

$i_L = i_{R_2} + i_c \quad ; i_{R_2} = i_L - i_c$

KVL: $v_s - R_1 i_L - v_L - v_c = 0$

$v_c - R_2 \cdot i_{R_2} = 0$

$$v_L = L \cdot \frac{di_L}{dt}$$

$$i_c = C \frac{dv_c}{dt}$$

Hence: $R_1 i_L + L \cdot \dfrac{di_L}{dt} + v_c = v_s$

$v_c - R_2 \cdot (i_L - i_c) = 0$

$$L\cdot\frac{di_L}{dt}+R_1\cdot i_L+v_C=v_S$$

$$-R_2\cdot i_L+R_2\cdot \underbrace{i_C}_{C\frac{dv_C}{dt}}+v_C=0$$

State equations:

$$L\cdot\frac{di_L}{dt}+R_1\cdot i_L+v_C=v_S$$

$$R_2\cdot C\cdot\frac{dv_C}{dt}+v_C-R_2\cdot i_L=0$$

or:
$$\frac{di_L}{dt}+\frac{R_1}{L}\cdot i_L+\frac{1}{L}\cdot v_C=\frac{1}{L}\cdot v_S \qquad (1)$$

$$\frac{dv_C}{dt}+\frac{1}{R_2C}\cdot v_C-\frac{1}{C}\cdot i_L=0 \qquad (2)$$

2^u order differential equation:

From (2):
$$i_L=C\frac{dv_C}{dt}+\frac{1}{R_2}\cdot v_C$$

Substitute into (1) gives:

$$\frac{d}{dt}\cdot\left(C\cdot\frac{dv_C}{dt}+\frac{1}{R_2}\cdot v_C\right)+\frac{R_1}{L}\cdot\left(C\frac{dv_C}{dt}+\frac{1}{R_2}\cdot v_C\right)+\frac{1}{L}\cdot v_C=1/L\cdot v_S$$

$$C\cdot\frac{d^2v_C}{dt^2}+\frac{1}{R_2}\cdot\frac{dv_C}{dt}+\frac{R_1C}{L}\cdot\frac{dv_C}{dt}+\frac{R_1}{LR_2}\cdot v_C+\frac{1}{L}\cdot v_C=\frac{1}{L}\cdot v_S$$

$$\frac{d^2 v_c}{dt^2} + \left(\frac{1}{CR_2} + \frac{R_1}{L}\right)\frac{dv_c}{dt} + \left(\frac{R_1}{LCR_2} + \frac{1}{LC}\right)v_c = \frac{1}{LC} \cdot v_s$$

Substitute numerical values:

$$\frac{d^2 v_c}{dt^2} + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \cdot \frac{dv_c}{dt} + \left(\frac{1}{2} + \frac{1}{2}\right) \cdot v_c = \frac{1}{2} \cdot 1$$

Recall:
$v_s(t) = u(t)$
$v_s(t) = 1$
for $t > 0$

$$\frac{d^2 v_c}{dt^2} + \sqrt{2} \cdot \frac{dv_c}{dt} + v_c = 1/2$$

characteristic equation:

$$s^2 + \sqrt{2}s + 1 = 0$$

$$s_{1/2} = -\frac{\sqrt{2}}{2} \pm \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 - 1}$$

$$s_{1/2} = -\frac{\sqrt{2}}{2} \pm \sqrt{\frac{1}{2} - 1}$$

$$s_{1/2} = -\frac{\sqrt{2}}{2} \pm j \cdot \frac{\sqrt{2}}{2} \quad ; \quad s_{1/2} = \frac{-1 \pm j}{\sqrt{2}}$$

$$v_c(t) = v_{ch}(t) + v_{cp}(t)$$

$$v_{ch}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$$

or

$$v_{ch}(t) = K_1 e^{\frac{-1+j}{\sqrt{2}}t} + K_2 e^{\frac{-1-j}{\sqrt{2}}t}$$

$$v_{ch}(t) = e^{-t/\sqrt{2}} \cdot \left(k_1 e^{jt/\sqrt{2}} + k_2 e^{-jt/\sqrt{2}}\right)$$

$$v_{ch}(t) = e^{-t/\sqrt{2}} \cdot \left(k_1 \cos t/\sqrt{2} + j \cdot k_1 \sin t/\sqrt{2} + k_2 \cos t/\sqrt{2} - j k_2 \sin t/\sqrt{2}\right)$$

$$v_{ch}(t) = e^{-t/\sqrt{2}} \cdot \left(A \cos t/\sqrt{2} + B \sin t/\sqrt{2}\right)$$

$v_{cp}(t) = K$ (constant because $v_s(t) = 1 \times u(t)$)

$$\frac{dv_{cp}(t)}{dt} = 0; \quad \frac{d^2 v_{cp}(t)}{dt^2} = 0$$

From the 2^{nd} order differential equation:

$$0 + \sqrt{2} \times 0 + A = 1/2$$

$$A = 1/2$$

Hence:

$$v_c(t) = e^{-t/\sqrt{2}} \cdot \left(A \cos t/\sqrt{2} + B \sin t/\sqrt{2}\right) + 1/2$$

$v_c(0-) = 0$ Hence, $A + 1/2 = 0$

$v_c(0+) = 0$ $\quad A = -1/2$

From the differential equation:

$$\frac{dv_c}{dt} + \frac{1}{R_2 C} \cdot v_c - \frac{1}{C} \cdot i_L = 0$$

$v_c(0-) = 0$

$i_L(0-) = 0$

$$v_c(0+) = 0; \; i_L(0+) = 0 \Rightarrow \left.\frac{dv_c}{dt}\right|_{0+} = 0$$

$$\frac{dv_c}{dt} = -\frac{1}{\sqrt{2}} e^{-\frac{t}{\sqrt{2}}} \left(A\cos\frac{t}{\sqrt{2}} + B\sin\frac{t}{\sqrt{2}}\right) +$$

$$+ e^{-t/\sqrt{2}} \cdot \left(A \cdot \frac{1}{\sqrt{2}} \cdot \left(-\sin\frac{t}{\sqrt{2}}\right) + B \cdot \frac{1}{\sqrt{2}} \cdot \cos\frac{t}{\sqrt{2}}\right) + 0$$

$$\left.\frac{dv_c}{dt}\right|_{0+} = -\frac{1}{\sqrt{2}}A + \frac{1}{\sqrt{2}}B$$

Since $\left.\frac{dv_c}{dt}\right|_{0+} = 0 \Rightarrow A = B$; Recall that $A = -1/2$

Hence, $B = -1/2$

$$v_c(t) = e^{-t/\sqrt{2}} \cdot \left(-\frac{1}{2}\cos\frac{t}{\sqrt{2}} - \frac{1}{2} \cdot \sin\frac{t}{\sqrt{2}}\right) + \frac{1}{2} \qquad t > 0$$

$$v_c(t) = -\frac{1}{2} e^{-t/\sqrt{2}} \cdot \left(\cos\frac{t}{\sqrt{2}} + \sin\frac{t}{\sqrt{2}}\right) + \frac{1}{2}$$

Another form:

$$\cos\frac{t}{\sqrt{2}} + \sin\frac{t}{\sqrt{2}} = V\sin\left(\frac{t}{\sqrt{2}} + \theta\right)$$

Find V and θ: $\qquad = V\sin\frac{t}{\sqrt{2}}\cos\theta + V \cdot \cos\frac{t}{\sqrt{2}}\sin\theta$

$\forall t$: $\quad 1 = V\cos\theta$; $1 = V \cdot \sin\theta$

$\tan\theta = 1$; $\theta = \pi/4$

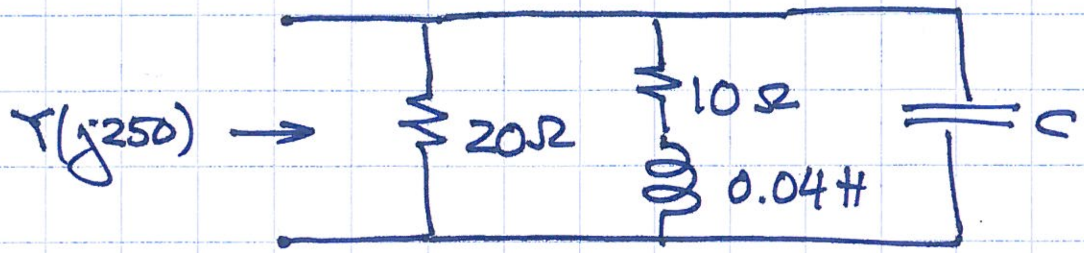
Hence, $V = 1/\cos\pi/4$; $V = \sqrt{2}$

$$v_c(t) = -\frac{1}{2} \cdot e^{-\frac{t}{\sqrt{2}}} \cdot \sqrt{2} \cdot \sin\left(\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right) + 1/2$$

$$v_c(t) = -\frac{\sqrt{2}}{2} e^{-t/\sqrt{2}} \cdot \sin\left(\frac{t}{\sqrt{2}} + \frac{\pi}{4}\right) + 1/2$$

for $t > 0$

Problem #4:



Let C = 0.6 mF ; $Y_{in}(j250) = ?$

$\omega = 250$ rad/s

$Y(j\omega) = ?$

Let:
$R_1 = 20\Omega$
$R_2 = 10\Omega$
$L = 0.04$ H
$C = 0.6\times 10^{-3}$

$$Y(j\omega) = \frac{1}{R_1} + \frac{1}{R_2 + j\omega L} + j\omega C$$

$$Y(j\omega) = \frac{1}{R_1} + \frac{R_2 - j\omega L}{R_2^2 + (\omega L)^2} + j\omega C$$

$$Y(j\omega) = \frac{1}{R_1} + \frac{R_2}{R_2^2 + (\omega L)^2} + j\left(\omega C - \frac{\omega L}{R_2^2 + (\omega L)^2}\right)$$

$$Y(j250) = \frac{1}{20} + \frac{10}{10^2 + (250\times 0.04)^2} + j\left(250\times 0.6\times 10^{-3} - \frac{250\times 0.04}{10^2 + (250\times 0.04)^2}\right)$$

$$Y(j250) = \frac{1}{20} + \frac{10}{100 + 100} + j\left(150\times 10^{-3} - \frac{10}{100+100}\right)$$

$$Y(j250) = \frac{1}{20} + \frac{10}{200} + j\left(0.15 - \frac{10}{200}\right)$$

$$Y(j250) = 1/10 + j\ \frac{30-10}{200}$$

$$Y(j250) = (0.1 + j0.1) \text{ siemens}$$

If $Y(j\omega)$ is to be real, then:

$$\omega C - \frac{\omega L}{R_2^2 + (\omega L)^2} = 0 \quad ; \omega = 250 \text{ rad/s}$$

or

$$C = \frac{L}{R_2^2 + (\omega L)^2}$$

$$C = \frac{0.04}{10^2 + (250 \times 0.04)^2}$$

$$C = \frac{0.04}{10^2 + 10^2} \quad ; C = \frac{0.04}{200}$$

$$C = 0.02 \times 10^{-2} \quad ; C = 0.2 \times 10^{-3}$$

$$C = 0.2 \text{ mF}$$