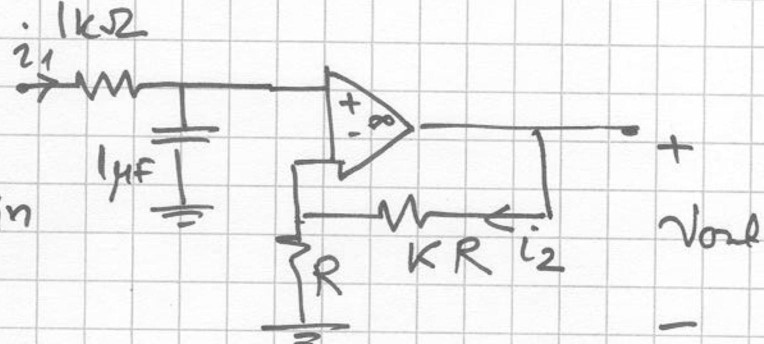
Final Exam: Fall '99



1 a)




$v_{in} = u(t)$ : unit step function

Ideal op-amp: $v_+ = v_-$

input current $= 0$

let $R_1 = 1k\Omega$

$C_1 = 1\mu F$

Then: $v_{in} = R_1 i_1 + v_+$

$v_+ = v_{C1}$

$i_1 = C_1 \dfrac{dv_{C1}}{dt}$

$$v_{in} = R_1 \cdot C_1 \cdot \frac{dv_{C1}}{dt} + v_{C1}$$

$$R_1 C_1 \cdot \frac{dv_{C1}}{dt} + v_{C1} = v_{in}$$

Solution: $R_1 C_1 \dfrac{dv_{C1}}{dt} + v_{C1} = 0$ ; $\dfrac{dv_{C1}}{dt} = -\dfrac{1}{R_1 C_1} v_{C1}$

$$v_{C_1}^{h} = K_1 e^{-t/R_1 C_1}$$

$$v_{C1}^{p} = A$$

$$A = 1 \quad \text{(unit step function)}$$

$$v_{C_1} = K_1 e^{-t/R_1C_1} + 1$$

$$v_{C_1}(0-) = v_{C_1}(0+)$$

$$v_{C_1}(0-) = 0$$

$$K_1 = -1$$

$$v_{C1} = 1 - e^{-t/R_1C_1}$$

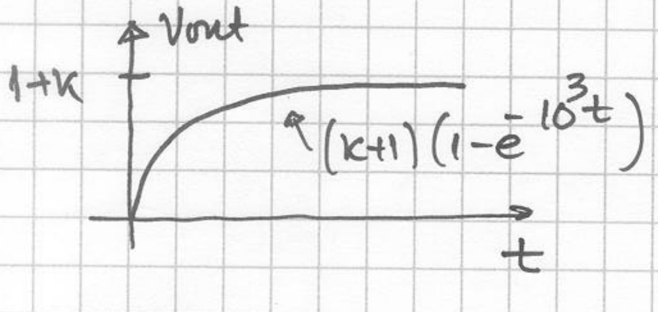
Then: $$v_{out} = R(1+K)\cdot i_2'$$

$$v_- = R\cdot i_2'$$

$$v_- = v_{C1}$$

$$i_2' = \frac{1}{R}\cdot v_{C1} \qquad R_1C_1 = 10^{-3}$$

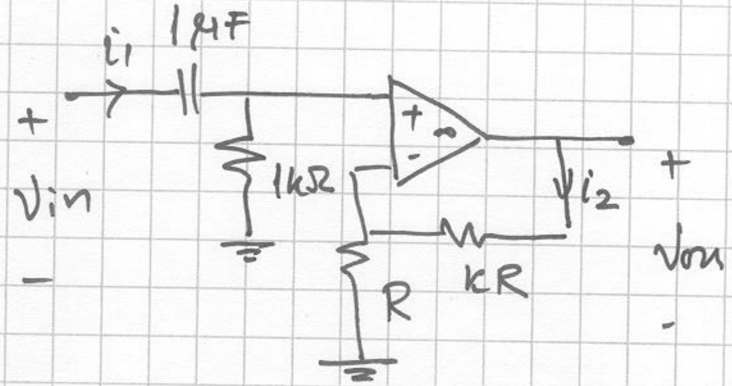
Hence: $$v_{out} = (1+K)\left(1 - e^{-t/R_1C_1}\right)$$




Low-pass filter.

1

b)




Again, as in 1 a):

$$v_{in} = v_{c1} + R_1 \cdot i_1$$

and $$v_{c1} = 1 - e^{-t/R_1C_1}$$

in this case:

$$V_{out} = (K+1) R \cdot i_2$$

$$R i_2 = V_-$$

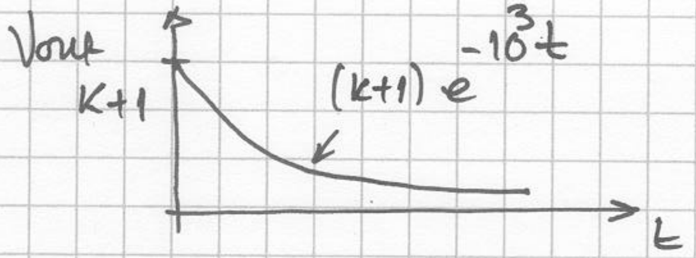
$$V_- = V_+$$

$$V_+ = R_1 i_1 \; ; \; i_1 = C_1 \frac{dV_{c1}}{dt}$$

$$R \cdot i_2 = R_1 \cdot C_1 \cdot \frac{dV_{c1}}{dt}$$

$$V_{out} = (K+1) \cdot R_1 \cdot C_1 \cdot \frac{dv_{c1}}{dt}$$

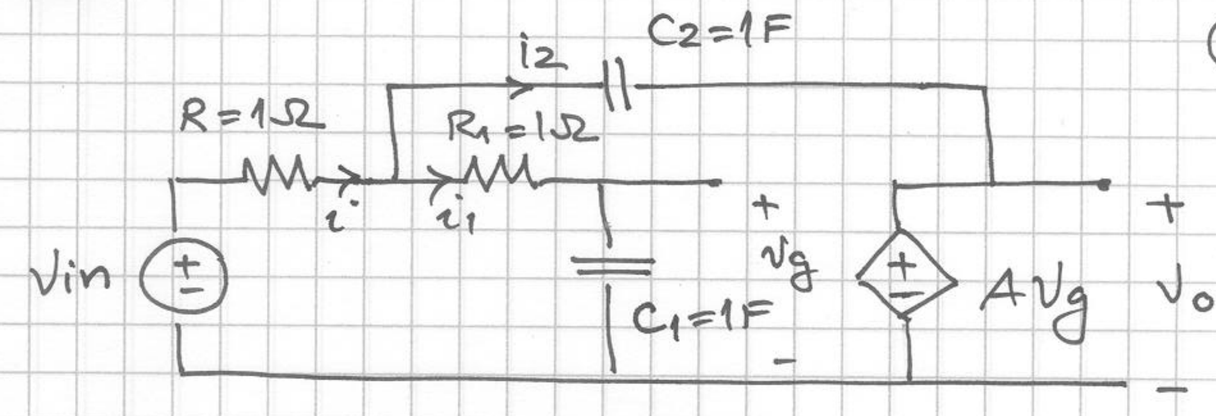
$$V_{out} = (K+1) \cdot e^{-t/R_1C_1}$$




$R_1 C_1 = 10^{-3}$

High-pass filter

(2)




State variables: $V_{C1}$

$V_{C2}$

$$R\cdot i + R_1\cdot i_1 + V_{C1} = V_{in}$$

$$R\cdot i + V_{C2} + A\cdot V_g = V_{in}$$

$$V_g = V_{C1}$$

$$i = i_1 + i_2$$

$$i_1 = C_1\cdot\frac{dV_{C1}}{dt}\ ;\ i_2 = C_2\cdot\frac{dV_{C2}}{dt}$$

Hence:

$$R\cdot\left(C_1\cdot\frac{dV_{C1}}{dt} + C_2\cdot\frac{dV_{C2}}{dt}\right) + R_1\cdot C_1\cdot\frac{dV_{C1}}{dt} + V_{C1} = V_{in}$$

$$R\cdot\left(C_1\cdot\frac{dV_{C1}}{dt} + C_2\cdot\frac{dV_{C2}}{dt}\right) + V_{C2} + A\cdot V_{C1} = V_{in}$$

$$(R+R_1)\cdot C_1\cdot\frac{dV_{C1}}{dt} + R\cdot C_2\cdot\frac{dV_{C2}}{dt} + V_{C1} = V_{in}$$

$$RC_1\cdot\frac{dV_{C1}}{dt} + RC_2\cdot\frac{dV_{C2}}{dt} + AV_{C1} + V_{C2} = V_{in}$$

Let $R = R_1 = 1\Omega$

$C_1 = C_2 = 1F$

$$2\frac{dv_{c1}}{dt} + \frac{dv_{c2}}{dt} + v_{c1} = v_{in}$$

$$\frac{dv_{c1}}{dt} + \frac{dv_{c2}}{dt} + A v_{c1} + v_{c2} = v_{in}$$

State equat:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{dv_{c1}}{dt} \\ \frac{dv_{c2}}{dt} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} \begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_{in}$$

or:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \; ; \; \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix} = \begin{pmatrix} 1-A & -1 \\ 2A-1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{dv_{c1}}{dt} \\ \frac{dv_{c2}}{dt} \end{pmatrix} + \begin{pmatrix} 1-A & -1 \\ 2A-1 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot v_{in}$$

Characterisk equat: $v = k e^{st}$

$$\det (s \cdot 1 + A) = 0$$

$$\det \begin{pmatrix} s+1-A & -1 \\ 2A-1 & s+2 \end{pmatrix} = 0$$

$$(s+1-A)(s+2)+2A-1=0$$

$$s^2+s-As+2s+2-2A+2A-1=0$$

$$s^2+3s-As+1=0$$

$$s^2+(3-A)s+1=0$$

$$s_{1/2}=-\frac{3-A}{2}\pm\sqrt{\left(\frac{3-A}{2}\right)^2-1}$$

System analysis:

a) over-damped system:

$s_{1/2}$ both real, distinct:

$$\left(\frac{3-A}{2}\right)^2-1>0 \quad \text{or} \quad \left(\frac{3-A}{2}+1\right)\left(\frac{3-A}{2}-1\right)>0$$

$$(5-A)(1-A)>0$$

$$(A-5)(A-1)>0$$

or: $A>5$
$A<1$

Note: $A<3$ required for $s_{1/2}$ to be negative.

Hence if $A<1$ system is over-damped

b) under-damped system:

$$(A-5)(A-1)<0$$

$$1<A<5$$

and $A<3$ (real part negative)

Hence: $1<A<3 \Rightarrow$ under-damped system

c) critically damped: $A=1$
$A=5$

and $A<3 \Rightarrow$

$A=1$ : critically damped system

d) undamped system: $A=3$

Output voltage:

$$V_o = A \cdot V_g$$

$$V_g = V_{C1}$$

$$V_o = A \cdot V_{C1}$$

From our state equation:

$$\left.\begin{aligned} \frac{dV_{C1}}{dt} + (1-A)\cdot V_{C1} - V_{C2} = 0 \\ \frac{dV_{C2}}{dt} + (2A-1)\cdot V_{C1} + 2V_{C2} = V_{in} \end{aligned}\right\}$$

Solution for $V_{C1} = \underbrace{K_1 e^{S_1 t} + K_2 e^{S_2 t}}_{\text{homogen solut.}} + V_1$ ← particula solut

But, for $A=1$

$$s^2 + 2s + 1 = 0 \qquad S_{1/2} = -1$$

critical case

Hence:

$$V_{C1} = K_1 e^{-t} + K_2 t e^{-t} + V$$

let us find $V$ first:

$V = \text{constant}$, because $v_{in} = \text{constant}$

$$(1-A)\cdot V_1 - V_2 = 0$$

particular solut. for $v_{c2}$

$$(2A-1)\cdot V_1 + 2V_2 = V_{in}$$

$A = 1 \Rightarrow V_2 = 0$

$\Rightarrow V_1 = V_{in}$ (constant)

Hence:

$$v_{c1} = (K_1 + K_2 t)\, e^{-t} + V_{in}$$

$v_{c1}(0-) = 0 \Rightarrow K_1 + V_{in} = 0 \quad K_1 = -V_{in}$

$v_{c1}(0+) = 0$

Then:

$$\frac{dv_{c1}}{dt} = K_2 e^{-t} - (K_1 + K_2 t)\, e^{-t}$$

From:

$$\frac{dv_{c1}}{dt} + (1-A)\cdot v_{c1} - v_{c2} = 0 \quad ; A = 1$$

$$\left(\frac{dv_{c1}}{dt}\right)_{0+} - v_{c2}(0+) = 0 \Rightarrow \left(\frac{dv_{c1}}{dt}\right)_{0+} = 0$$

or:

$$K_2 - K_1 = 0 \Rightarrow K_2 = K_1$$

$$K_2 = -V_{in}$$

Hence:

$$v_{C1} = \left(-V_{in} - V_{in}\cdot t\right)e^{-t} + V_{in}$$

$$v_{C1} = V_{in}\left(1 - (1+t)e^{-t}\right)$$

$$v_o = A\cdot V_{in}\left(1 - (1+t)e^{-t}\right)$$

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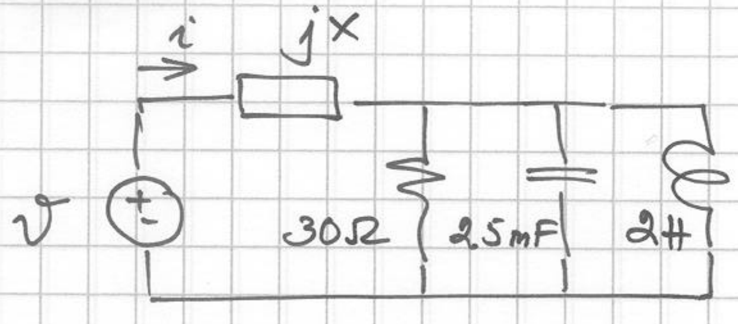


$W = 10$

$V = 96 \cos(10t + \pi/3)$

input impedance: $R = 30\Omega$ ; $X_C = \frac{1}{WC}$ ; $X_L = WL$

$$Y = \frac{1}{R} + j\omega C + \frac{1}{j\omega L}$$

$$Y = \frac{1}{30} + j \cdot 10 \cdot 2.5 \times 10^{-3} + \frac{1}{j \times 10 \cdot 2}$$

$$Y = \frac{1}{30} + 25 \times 10^{-3} j + \frac{1}{20j}$$

$$Z = \frac{1}{Y}$$

$$Z = \frac{1}{\frac{1}{30} + j\left(25 \times 10^{-3} - \frac{1}{20}\right)}$$

$$Z = \frac{1}{\frac{1}{30} + j \cdot \frac{500 \times 10^{-3} - 1}{20}}$$

$$Z = \frac{1}{1/30 + j \cdot \frac{-0.5}{20}}$$

$$Z = \frac{1}{\frac{1}{30} - \frac{j}{40}}$$

$$Z = \frac{1}{\frac{4-j3}{120}}$$

$$Z = \frac{120}{4-j3} \; ; \quad \frac{120(4+j3)}{16+9}$$

$$Z = \frac{120}{25}\cdot(4+j3)$$

Hence:

$$Z_{total} = jX + \frac{120}{25}\cdot(4+j3)$$

and $X + \frac{120}{25}\times 3 = 0$ ; $X = -\frac{360}{25}$

$$X = -\frac{72}{5}$$

$X = -14.4$ (reactance)

$\frac{1}{WC} = 14.4$

Capacitor

(otherwise inductor would have negative inductance)

$C = \frac{1}{144}$ F

$$\underline{V} = 96\, e^{j\pi/3}$$

$$\underline{I} = \underline{V} / Z_{total}$$

Where $Z_{TOTAL} = \dfrac{120 \times 4}{25}$

$$Z_{total} = \frac{96}{5}$$

$$\underline{I} = \frac{96}{96/5} \cdot e^{j\pi/3}$$

$$\underline{I} = 5 e^{j\pi/3}$$

$$i(t) = 5 \cos(10t + \pi/3)$$

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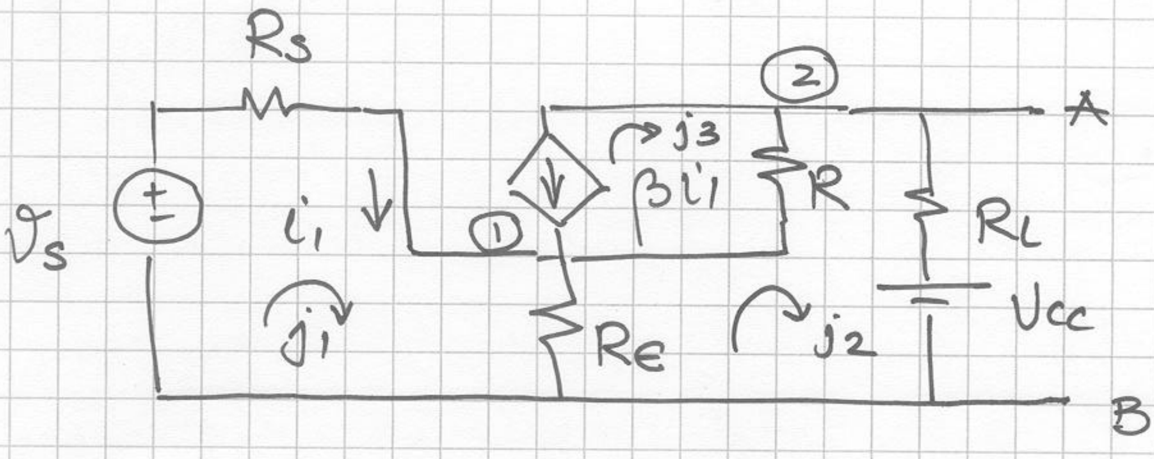


a)

Nodal equations:

$$\frac{v_1 - V_S}{R_S} + \frac{V_1}{R_E} + \frac{V_1 - V_2}{R} - \beta i_1 = 0$$

$$\beta i_1 + \frac{V_2 - V_1}{R} + \frac{V_2 - V_{CC}}{R_L} = 0$$

$$i_1 = \frac{V_S - V_1}{R_S}$$

Hence:
$$\frac{V_1 - V_S}{R_S} + \frac{V_1}{R_E} + \frac{V_1 - V_2}{R} - \beta \cdot \frac{V_S - V_1}{R_S} = 0$$

$$\beta \cdot \frac{V_S - V_1}{R_S} + \frac{V_2 - V_1}{R} + \frac{V_2 - V_{CC}}{R_L} = 0$$

$$v_1 \cdot \left( \frac{1+\beta}{R_S} + \frac{1}{R_E} + \frac{1}{R} \right) - v_2 \cdot \frac{1}{R} = \frac{1+\beta}{R_S} \cdot V_S$$

$$-\frac{1+\beta}{R} \cdot v_1 + v_2 \cdot \left( \frac{1}{R} + \frac{1}{R_L} \right) = -\frac{\beta}{R_S} \cdot V_S + \frac{1}{R_L} \cdot V_{CC}$$

Validity can be checked by setting $\beta = 0$
Nodal equations should be symmetric:

$$\left(\frac{1}{R_S} + \frac{1}{R_E} + \frac{1}{R}\right) v_1 - \frac{1}{R} \cdot v_2 = \frac{1}{R_S} \cdot v_S$$

$$-\frac{1}{R} \cdot v_1 + \left(\frac{1}{R} + \frac{1}{R_L}\right) \cdot v_2 = \frac{1}{R_L} \cdot v_{CC}$$

✓

b) Mesh equations:

$$v_S - (R_S + R_E) \cdot j_1 + R_E \cdot j_2 = 0$$

$$j_3 = -\beta \cdot i_1$$

$$V_{CC} + (R_L + R + R_E) j_2 - R_E \cdot j_1 - R \cdot j_3 = 0$$

Since $\quad i_1 = j_1$

$$\Rightarrow j_3 = -\beta \cdot j_1$$

and

$$(R_S + R_E) \cdot j_1 - R_E \cdot j_2 = V_S$$

$$-R_E \cdot j_1 - R \cdot (-\beta \cdot j_1) + (R_L + R + R_E) j_2 = -V_{CC}$$

or:

$$(R_S + R_E) j_1 - R_E \cdot j_2 = V_S$$

$$-(R_E + \beta R) j_1 + (R + R_E + R_L) j_2 = -V_{CC}$$

Validity check: $\beta = 0$

$$(R_S + R_E)\, j_1 - R_E\, j_2 = V_S$$

$$-R_E\, j_1 + (R + R_E + R_L)\, j_2 = -V_{CC}$$

symmetric.

Let $R = R_S = R_E = R_L$

Thevenin's voltage: (AB open circuited)

From nodal equations:

$$\frac{1}{R}\cdot(3+\beta)V_1 - V_2 \cdot \frac{1}{R} = \frac{1+\beta}{R}\cdot V_S$$

$$-\frac{1+\beta}{R}\cdot V_1 + \frac{2}{R}\cdot V_2 = -\frac{\beta}{R}\cdot V_S + \frac{1}{R} V_{CC}$$

$V_{AB} = V_2$

$$(3+\beta)V_1 - V_2 = (1+\beta)V_S$$

$$-(1+\beta)V_1 + 2V_2 = -\beta V_S + V_{CC}$$

Solving:

$$-\frac{V_2 + (1+\beta)V_S}{3+\beta}\cdot(1+\beta) + 2V_2 = -\beta V_S + V_{CC}$$

$$V_2 \cdot \left(2 - \frac{1+\beta}{3+\beta}\right) = -\beta V_S + \frac{(1+\beta)^2}{3+\beta} \cdot V_S + V_{CC}$$

$$V_2 \cdot (6+2\beta-1-\beta) = \left(-\beta(3+\beta)+(1+\beta)^2\right) V_S$$
$$+ (3+\beta) \cdot V_{CC}$$

$$V_2 = \frac{(1-\beta) V_S + (3+\beta) V_{CC}}{5+\beta}$$

Thévenin's voltage

Thévenin's resistance

All independent sources removed


$$R_{th} = V_x / I_x$$

$$j_1 \cdot (R_S + R_E) - R_E j_2 = 0$$

$$j_3 = -\beta \cdot i_1$$

$$-R_E \cdot j_1 + (R_E + R + R_L) j_2 - R j_3 - R_L \cdot j_4 = 0$$

$$V_x - R_L \cdot j_4 + R_L \cdot j_2 = 0$$

Also: $i_1 = j_1$

Then:

$$(R_S + R_E)\, j_1 - R_E \cdot j_2 = 0$$

$$j_3 = -\beta \cdot j_1$$

$$-R_E \cdot j_1 + (R_E + R + R_L)\, j_2 - R \cdot j_3 - R_L\, j_4 = 0$$

$$R_L (j_4 - j_2) = V_x$$

Let $R = R_S = R_E = R_L$

then:

$$2 j_1 - j_2 = 0$$

$$j_3 = -\beta \cdot j_1$$

$$-j_1 + 3 j_2 - j_3 - j_4 = 0$$

$$R (j_4 - j_2) = V_x$$

We only need $V_x / j_4$ (since $I_x = j_4$)

$$j_2 = 2 j_1$$

Then: $-j_1 + 6 j_1 - j_3 - j_4 = 0$

$$5 j_1 + \beta j_1 = j_4 \Rightarrow j_1 = \frac{j_4}{5+\beta}$$

$$j_2 = \frac{2}{5+\beta} \cdot j_4$$

then: $V_x = R \cdot \left( j_4 - \dfrac{2 j_4}{5+\beta} \right)$

$$R_{th} = \frac{V_x}{j_4} \; ; \; R_{th} = \frac{3+\beta}{5+\beta} R$$

Test: let $\beta = 0$


$$\leftarrow \frac{\left(\frac{R \cdot R}{R+R} + R\right) R}{\frac{R \cdot R}{R+R} + R + R} = \frac{\frac{3R}{2} \cdot R}{\frac{3R}{2} + R}$$

$$= \frac{3}{5} R$$