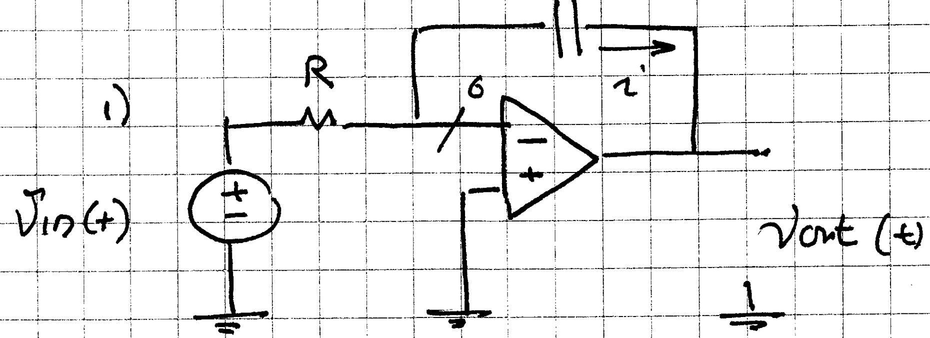
Midterm exam: #1

1)



$$v_{in} = R \cdot i$$

$$i = -C \cdot \frac{dV_{out}}{dt}$$

$$V_{in} = -RC \cdot \frac{dV_{out}}{dt}$$

$$V_{out} = -\frac{1}{RC}\int_0^t V_{in} \cdot dt$$

This is an integrator circuit

If $v_{in}(t) = \cos(250t)$

$R = 4\,k\Omega$

$C = 1\,\mu F \qquad RC = 4\times10^{-3}\,s$

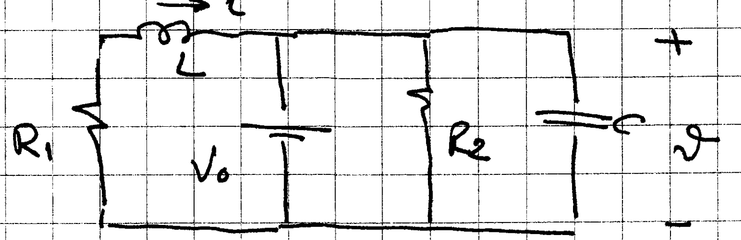
$$V_{out} = -\frac{1}{4\times10^{-3}} \cdot \int_0^t (\cos(250t))\,dt$$

$$V_{out} = -\frac{1}{4\times10^{-3}} \cdot \frac{1}{250} \cdot \sin(250t)$$

$$V_{out} = -\sin(250t)$$

(3)

2. Switch closed



$V_0 = 10\,V$
$R_1 = 20\,\Omega$
$R_2 = 0.5\,\Omega$
$L = 1\,H$
$C = 0.25\,F$

at $t = 0_-$:

$v(0_-) = 10\,V$ All units in SI

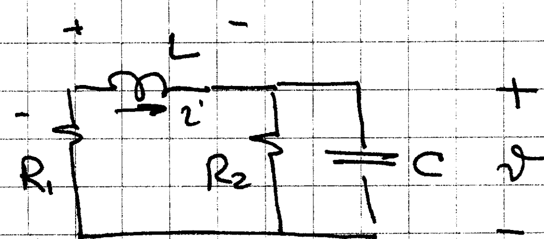
$i_{(0_-)} = -\dfrac{V_0}{R_1}$

$i_{(0_-)} = -\dfrac{10}{20}$; $i(0_-) = -0.5\,A$

$v(0+) = 10\,V$

$i(0+) = -0.5\,A$

Switch open:



$$v + L\frac{di}{dt} + R_1 i = 0$$

$$i = \frac{v}{R_2} + C\cdot\frac{dv}{dt}$$

$$\left.\begin{aligned} L\frac{di}{dt} + R_1 i + v &= 0 \\ C\frac{dv}{dt} - i + \frac{1}{R_2}\cdot v &= 0 \end{aligned}\right\}$$

$$\frac{di}{dt} + \frac{R_1}{L} \cdot i + \frac{1}{L} \cdot v = 0$$

$$\frac{dv}{dt} - \frac{1}{C} \cdot i + \frac{1}{R_2 C} v = 0$$

$$\begin{pmatrix} di/dt \\ dv/dt \end{pmatrix} + \begin{pmatrix} R_1/L & 1/L \\ -1/C & \frac{1}{R_2 C} \end{pmatrix} \cdot \begin{pmatrix} i \\ v \end{pmatrix} = 0$$

or

$$\frac{d}{dt}\begin{pmatrix} i \\ v \end{pmatrix} + \begin{pmatrix} R_1/L & 1/L \\ -1/C & \frac{1}{R_2 C} \end{pmatrix} \begin{pmatrix} i \\ v \end{pmatrix} = 0$$

Convert it into a 2nd order different equat.

$$v = -L\frac{di}{dt} - R_1 i$$

From $\frac{dv}{dt} - \frac{1}{C} i + \frac{1}{R_2 C} v = 0$

We obtain:

$$\frac{d}{dt}\left(-L\frac{di}{dt} - R_1 i\right) - \frac{1}{C} \cdot i + \frac{1}{R_2 C} \cdot \left(-L\frac{di}{dt} - R_1 i\right) = 0$$

$$L \cdot \frac{d^2 i}{dt^2} + R_1 \cdot \frac{di}{dt} + \frac{1}{C} \cdot i + \frac{L}{R_2 C} \cdot \frac{di}{dt} + \frac{R_1}{R_2 C} \cdot i = 0$$

$$\frac{d^2 i}{dt^2} + \left(\frac{R_1}{L} + \frac{1}{R_2 C}\right) \cdot \frac{di}{dt} + \left(\frac{1}{LC} + \frac{R_1}{R_2 LC}\right) i = 0$$

or, finally

$$\frac{d^2 i}{dt^2} + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)\cdot\frac{di}{dt} + \frac{1}{LC}\cdot\left(1+\frac{R_1}{R_2}\right)i = 0$$

$$s^2 + \left(\frac{R_1}{L} + \frac{1}{CR_2}\right)s + \frac{1}{LC}\cdot\left(1+\frac{R_1}{R_2}\right) = 0$$

$R_1 = 20$; $R_2 = 0.5$
$L = 1$
$C = 0.25$

$$s^2 + \left(\frac{20}{1} + \frac{1}{0.25\times 0.5}\right)s + \frac{1}{1\times 0.25}\cdot\left(1+\frac{20}{0.5}\right) = 0$$

$$s^2 + 28s + 164 = 0$$

$$s_{1/2} = -14 \pm \sqrt{196-164}$$

$$s_{1/2} = -14 \pm \sqrt{32}$$

$$s_{1/2} = -14 \pm 4\sqrt{2}$$ Two real roots

$$i(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$

$i(0+) = -0.5 \Rightarrow K_1 + K_2 = -0.5$

$$\left(\frac{di}{dt}\right)_{0+} = K_1 S_1 + K_2 S_2$$

Note from: $\left(\frac{di}{dt}\right) + 20i + V = 0$

$\left.\frac{di}{dt}\right|_{0+} = -20\cdot i(0+) - V(0+)$; $\left.\frac{di}{dt}\right|_{0+} = -20\cdot(-0.5) - 10$

$\left(\frac{di}{dt}\right)\Big|_{0+} = 0$

$k_1 + k_2 = -0.5$ $\qquad k_2 = -k_1 \cdot \frac{s_1}{s_2}$

$k_1 s_1 + k_2 s_2 = 0$

$k_1 \left(1 - \frac{s_1}{s_2}\right) = -\frac{1}{2}$

$k_1 = -\frac{1}{2} \cdot \frac{s_2}{s_2 - s_1}$ $\qquad s_{1/2} = -14 \pm 4\sqrt{2}$

$s_1 = -14 + 4\sqrt{2}$

$s_2 = -14 - 4\sqrt{2}$

$k_2 = \frac{1}{2} \cdot \frac{s_2}{s_2 - s_1} \cdot \frac{s_1}{s_2}$

$k_2 = \frac{1}{2} \cdot \frac{s_1}{s_2 - s_1}$

Hence, $s_2 - s_1 = -8\sqrt{2}$

$k_1 = -\frac{1}{2} \cdot \frac{-14 - 4\sqrt{2}}{-8\sqrt{2}}$; $k_1 = -\frac{7 + 2\sqrt{2}}{8\sqrt{2}}$

$k_2 = \frac{1}{2} \cdot \frac{-14 + 4\sqrt{2}}{-8\sqrt{2}}$; $k_2 = \frac{7 - 2\sqrt{2}}{8\sqrt{2}}$

Alternah.: $k_1 = -\frac{7\sqrt{2} + 4}{8 \times 2}$; $k_1 = -\frac{7\sqrt{2} + 4}{16}$

$k_2 = \frac{7\sqrt{2} - 4}{8 \times 2}$ $\qquad k_2 = \frac{7\sqrt{2} - 4}{16}$

3. a) $f(t) = te^{-at}$

There are severa ways to solve this proble.

$f_1(t) = e^{-at}$

$f_2(t) = t$

$f_3(t) = 1$

Then: $F_3(s) = \frac{1}{s}$

$F_2(s) = \frac{1}{s^2}$

$F_1(s) = \frac{1}{s+a}$

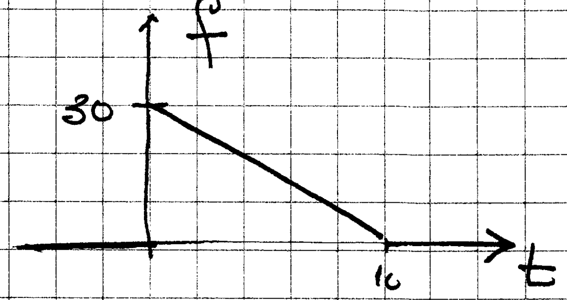
$$F(s) = \frac{1}{(s+a)^2}$$

b) $f(t) = e^{-at} \sin \omega t$

$f_1(t) = \sin \omega t \qquad F_1(s) = \frac{w}{s^2+w^2}$

$$F(s) = \frac{w}{(s+a)^2 + w^2}$$

c)



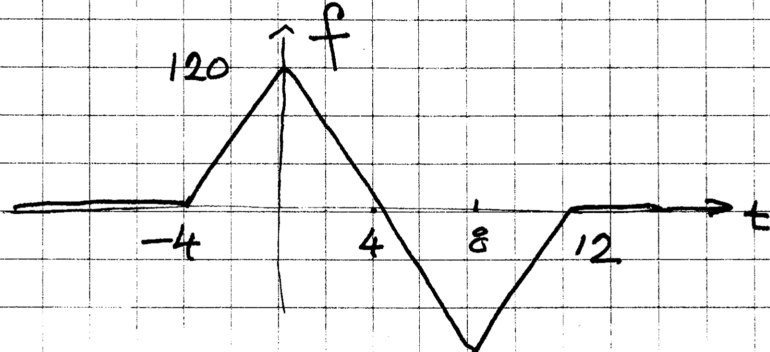
$$f(t) = -\frac{30}{10}\cdot t\cdot\left(u(t) - u(t-10)\right)$$

$$f(t) = -3t\,u(t) + 3t\,u(t-10)$$

$$f(t) = -3t\,u(t) + 3(t-10)\cdot u(t-10) + 30\,u(t-10)$$

$$F(s) = -\frac{3}{s^2} + \frac{3}{s^2}e^{-10s} + \frac{30}{s}e^{-10s}$$

d)



$$f(t) = 30(t+4)\cdot[u(t+4) - u(t)]$$
$$-30(t-4)[u(t) - u(t-8)]$$
$$+30(t-12)[u(t-8) - u(t-12)]$$

$$f(t) = 30(t+4)u(t+4) - 30(t+4)u(t)$$
$$-30(t-4)u(t) + 30(t-4)u(t-8)$$
$$+30(t-12)u(t-8) - 30(t-12)u(t-12)$$

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$$f(t) = 30(t+4)u(t+4) - 30t\,u(t) - \cancel{120u(t)}$$
$$-30t\,u(t) + \cancel{120u(t)} + 30(t-8)u(t-8) + \cancel{120u(t-8)}$$
$$+30(t-8)u(t-8) - \cancel{120u(t-8)} - 30(t-12)u(t-12)$$

$$f(t) = 30(t+4)u(t+4) - 60t\,u(t) +$$
$$+60(t-8)u(t-8) - 30(t-12)u(t-12)$$

$$F(s) = \frac{30}{s^2}e^{4s} - \frac{60}{s^2} + \frac{60}{s^2}e^{-8s} - \frac{30}{s^2}e^{-12s}$$

4.

a. $F(s) = \dfrac{3s+1}{(s+2)(s^2+4s+8)}$

b. $F(s) = \dfrac{s(s+2)\,e^{-3s}}{(s+1)^2 \cdot (s+4)}$

a) $F(s) = \dfrac{3s+1}{(s+2)(s^2+4s+8)}$

$$= \frac{A}{s+2} + \frac{Bs+C}{s^2+4s+8}$$

$$A = \lim_{s \to -2} (s+2) \cdot F(s)$$

$$A = \lim_{s \to -2} \frac{3s+1}{s^2+4s+8}$$

$$A = \frac{-6+1}{4-8+8}$$

$$A = -\frac{5}{4}$$

Hence: $F(s) = -\dfrac{5}{4} \cdot \dfrac{1}{s+2} + \dfrac{Bs+C}{s^2+4s+8}$

Hence: $\dfrac{Bs+C}{s^2+4s+8} = \dfrac{3s+1}{(s+2)(s^2+4s+8)} + \dfrac{5}{4} \cdot \dfrac{1}{s+2}$

Note that

$$\frac{3s+1}{(s+2)(s^2+4s+8)} + \frac{5}{4(s+2)}$$

$$= \frac{4(3s+1)+5\cdot(s^2+4s+8)}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s^2+32s+44}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s(s+2)+22(s+2)}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s+22}{4(s^2+4s+8)}$$

$$= \frac{5s+22}{4[(s+2)^2+4]}$$

$$= \frac{5(s+2)}{4[(s+2)^2+4]} + \frac{12}{4[(s+2)^2+4]}$$

$$F(s) = -\frac{5}{4(s+2)} + \frac{5(s+2)}{4[(s+2)^2+4]} + \frac{3}{(s+2)^2+4}$$

$$f(t) = \left\{-\frac{5}{4}e^{-2t} + \frac{5}{4}\cos 2t\, e^{-2t} + \frac{3}{2}\sin 2t\, e^{-2t}\right\}u(t)$$

$$f(t) = \frac{1}{4}\left\{-5 + 5\cos 2t + 6\sin 2t\right\}e^{-2t}u(t)$$

b) $$F(s) = \frac{s(s+2)e^{-3s}}{(s+1)^2 \cdot (s+4)}$$

$$F_1(s) = \frac{s(s+2)}{(s+1)^2 \cdot (s+4)}$$

$$= \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+4}$$

$$C = \lim_{s \to -4} (s+4) F_1(s)$$

$$= \frac{-4 \cdot (-4+2)}{(-4+1)^2}$$

$$C = \frac{8}{9}$$

$$B = \lim_{s \to -1} (s+1)^2 F_1(s)$$

$$= \frac{-1 \cdot (-1+2)}{-1+4}$$

$$B = -1/3$$

$$A = \lim_{s \to -1} \frac{d}{ds} (s+1)^2 \cdot F_1(s)$$

$$= \lim_{s \to -1} \cdot \frac{d}{ds} \cdot \frac{s(s+2)}{s+4}$$

$$= \lim_{s \to -1} \cdot \frac{(2s+2) - s(s+2)}{(s+4)^2} \quad ; \quad A = \frac{-2+2+1(-1+2)}{(-1+4)^2}$$

$$A = \frac{1}{9}$$

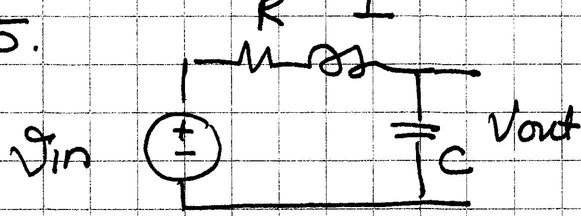
$$F_1(s) = \frac{1}{9}\cdot\frac{1}{s+1} - \frac{1}{3}\cdot\frac{1}{(s+1)^2} + \frac{8}{9}\cdot\frac{1}{s+4}$$

$$f_1(t) = \left\{\frac{1}{9}e^{-t} - \frac{1}{3}te^{-t} + \frac{8}{9}e^{-4t}\right\}u(t)$$

$$F(s) = e^{-3s}\cdot F_1(s)$$

$$f(t) = \left\{\frac{1}{9}e^{-(t-3)} - \frac{1}{3}(t-3)e^{-(t-3)} + \frac{8}{9}e^{-4(t-3)}\right\}u(t-3)$$

5.



$R = 4\Omega$
$L = 1H$
$C = 0.2F$

$$V_{out} = \frac{1}{Cs} \cdot \left(R + Ls + \frac{1}{Cs}\right) V_{in}$$

$$H(s) = \frac{1}{LCs^2 + RCs + 1}$$

$$H(s) = \frac{1}{0.2s^2 + 0.8s + 1}$$

$$H(s) = \frac{5}{s^2 + 4s + 5}$$

impulse response: $V_{out}(s) = H(s)$ $\qquad v_{in}(t) = \delta(t)$, $V_{in}(s) = 1$

$$V_{out}(s) = \frac{5}{(s+2)^2 + 1}$$

$$V_{out}(t) = 5 \sin t \cdot e^{-2t}\, u(t)$$

Step respons: $V_{out}(s) = \frac{5}{s^2+4s+5} \cdot \frac{1}{s}$ $\qquad V_{in}(t) = u(t)$, $V_{in}(s) = \frac{1}{s}$

$$V_{out}(s) = \frac{A}{s} + \frac{Bs + C}{(s+2)^2 + 1}$$

$$A = \lim_{s \to 0} s \cdot \frac{5}{s(s^2+4s+5)} \; ; \; A = 1$$

$$\frac{5}{s[(s+2)^2+1]} - \frac{1}{s} = \frac{5 - (s+2)^2 - 1}{s[(s+2)^2+1]} = \frac{4 - s^2 - 4s - 4}{s[(s+2)^2+1]} = -\frac{s+4}{(s+2)^2+1}$$

Note that:

$$-\frac{s+4}{(s+2)^2+1} = -\frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$

Hence:

$$V_{out}(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+1} - \frac{2}{(s+2)^2+1}$$

$$V_{out}(t) = \left(1 - \cos t\, e^{-2t} - 2\sin t\, e^{-2t}\right) u(t)$$

Check: since $\delta(t) = \frac{du(t)}{dt}$

Impulse response $= \frac{d}{dt}$ (step response)

$$\frac{dV_{out}}{dt} = \left(\sin t\, e^{-2t} + 2\cos t\, e^{-2t} - 2\cos t\, e^{-2t} + 4\sin t\, e^{-2t}\right) u(t)$$

$$\frac{dV_{out}}{dt} = 5\sin t\, e^{-2t}\, u(t)$$

As expected. ✓