Midterm Exam: Summer 2011 ①

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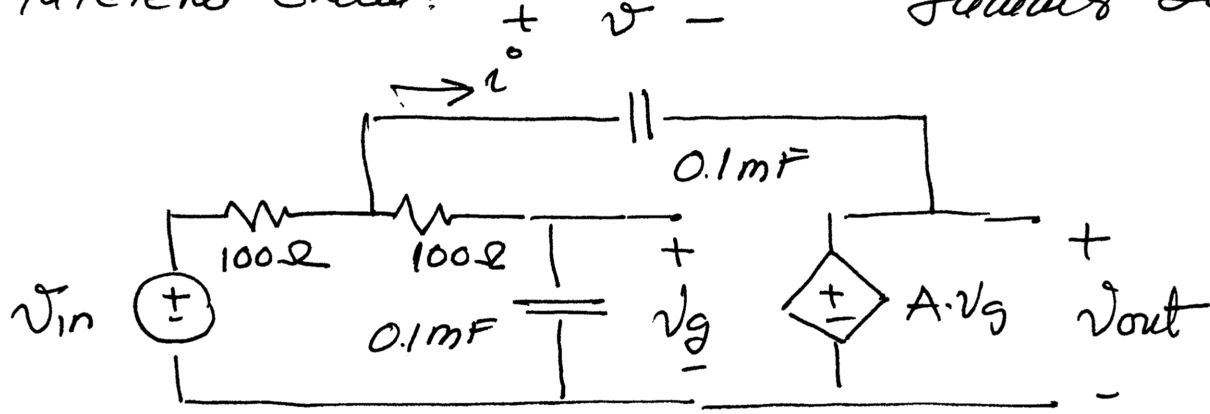

State variables: v ; v_g

Constitutive relationships:

$$i = 10^{-4}\cdot\frac{dv}{dt}$$

$$i_{in} - i = 10^{-4}\cdot\frac{dv_g}{dt}$$

KCL:

Hence, $i_{in} = 10^{-4}\cdot\left(\frac{dv}{dt} + \frac{dv_g}{dt}\right)$

Time-domain equation (KVL):

$$v_{in} - 100\times i_{in} - 100\cdot(i_{in} - i) - v_g = 0$$

$$v_{in} - 100\times i_{in} - v - A\cdot v_g = 0$$

Hence: $200\times i_{in} - 100\times i + v_g = v_{in}$

$$100\times i_{in} + v + A\cdot v_g = v_{in}$$

$$\left.\begin{aligned} &10^{-2}\cdot\frac{dv}{dt}+2\times 10^{-2}\cdot\frac{dv_g}{dt}+v_g=v_{in}\\ &10^{-2}\cdot\frac{dv}{dt}+10^{-2}\cdot\frac{dv_g}{dt}+v+A\,v_g=v_{in}\end{aligned}\right\}$$

Or:

$$\frac{dv}{dt}+2\times\frac{dv_g}{dt}+100\cdot v_g=100\times v_{in} \qquad (1)$$

$$\frac{dv}{dt}+\frac{dv_g}{dt}+100\cdot v+100\times A\times v_g=100\times v_{in} \qquad (2)$$

State equations:

After substracting (2) from (1)

$$\frac{dv_g}{dt}+100\times(v_g-v)-100\cdot A\cdot v_g=0$$

or:

$$\frac{dv_g}{dt}+100\,(1-A)\cdot v_g-100\,v=0 \qquad (3)$$

Hence: $v=\frac{1}{100}\times\frac{dv_g}{dt}+(1-A)\cdot v_g$

and $\frac{dv}{dt}=\frac{1}{100}\times\frac{d^2v_g}{dt^2}+(1-A)\cdot\frac{dv_g}{dt}$

Substituting into (1) gives:

$$\frac{1}{100}\times\frac{d^2v_g}{dt^2}+(1-A)\cdot\frac{dv_g}{dt}+2\cdot\frac{dv_g}{dt}+100\cdot v_g=100\times v_{in}$$

or:
$$\frac{d^2 v_g}{dt^2} + 100(1-A)\cdot\frac{dv_g}{dt} + 200\times\frac{dv_s}{dt} + 10^4\cdot v_g = 10^4\cdot v_{in}$$

Finally:
$$\frac{d^2 v_g}{dt^2} + 100(3-A)\frac{dv_g}{dt} + 10^4\cdot v_g = 10^4\times v_{in} \qquad (4)$$

2nd order equation in terms of v_g.

Characteristic equation:

$$s^2 + 100(3-A)s + 10^4 = 0 \qquad (5)$$

Zeros (natural frequency of the circuit):

$$s_{1/2} = -50(3-A) \pm \sqrt{(50(3-A))^2 - 10^4}$$

$$= 50(A-3) \pm 50\sqrt{(3-A)^2-4}$$

$$= 50(A-3) \pm 50\cdot\sqrt{A^2-6A+5}$$

Circuit is stable if $Re\{s_{1/2}\} < 0$

Stable system: $0 < A < 3$ (approximate solution)

Consider the expression:

$$\frac{D}{50} = A - 3 \pm \sqrt{A^2 - 6A + 5}$$

$$D/50 = A - 3 \pm \sqrt{(A-5)(A-1)}$$

There are four cases to consider:

- Overdamped: all zeros real

$$(A-5)(A-1) > 0$$

$$A > 5 \quad \text{(unstable)}$$

$$A < 1$$

$\rightarrow$ Hence: $A < 1$

- Underdamped: complex-conjugate zeros

$$A < 5$$

$$A > 1$$

with stability condition:

$$\rightarrow 1 < A < 3$$

- Critically damped: two real zeros

$$A = 1$$

$$A = 5$$

with stability condition.

$$\rightarrow A = 1$$

- Undamped: complex-conjugate (imaginary)

$$\rightarrow A = 3$$

Critically damped case: $A = 1$

Equation (4) becomes:

$$\frac{d^2 v_g}{dt^2} + 100 \times 2 \times \frac{dv_g}{dt} + 10^4 v_g = 10^4 \cdot v_{in}$$

$$\frac{d^2 v_g}{dt^2} + 200 \times \frac{dv_g}{dt} + 10^4 \cdot v_g = 10^4 \cdot v_{in}$$

characteristic equation:

$$s^2 + 200s + 10^4 = 0$$

$$s_{1/2} = -100$$

$$v_g = (k_1 + k_2 t)\, e^{-100t}\, u(t) + \text{particular solution that depends on } v_{in}.$$

$$v_{out} = A \cdot v_g$$

For $A = 1$: $v_{out} = v_g$

Addendum:

Matrix form of the state equation:

From (1) and (2):

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} dv/dt \\ dv_g/dt \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 100 & 100A \end{pmatrix}\begin{pmatrix} v \\ v_g \end{pmatrix} = 100\times\begin{pmatrix} 1 \\ 1 \end{pmatrix} v_{in}$$

Since: $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{1-2}\begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} dv/dt \\ dv_g/dt \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\times 100\times\begin{pmatrix} 0 & 1 \\ 1 & A \end{pmatrix}\begin{pmatrix} v \\ v_g \end{pmatrix} = 100\times\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} v_{in}$$

Let $x = \begin{pmatrix} dv/dt \\ dv_g/dt \end{pmatrix}$

$$\frac{dx}{dt} + 100\begin{pmatrix} 2 & -1+2A \\ -1 & 1-A \end{pmatrix}\cdot x = 100\begin{pmatrix} 1 \\ 0 \end{pmatrix} v_{in}$$

$$\det(\underbrace{sI + A}) = 0$$

$$s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 100\begin{pmatrix} 2 & -1+2A \\ -1 & 1-A \end{pmatrix} = 0$$

$$\det\begin{pmatrix} s+200 & 100(2A-1) \\ -100 & s+100(1-A) \end{pmatrix} = 0$$

$$(s+200)(s+100(1-A)) + 10^4\cdot(2A-1) = 0$$

$$s^2 + 200s + 100s(1-A) + 2\times10^4(1-A) + 10^4(2A-1) = 0$$

Or, finally;

$$s^2+100(3-A)s+10^4=0$$

Equivalent to the character equation (5).

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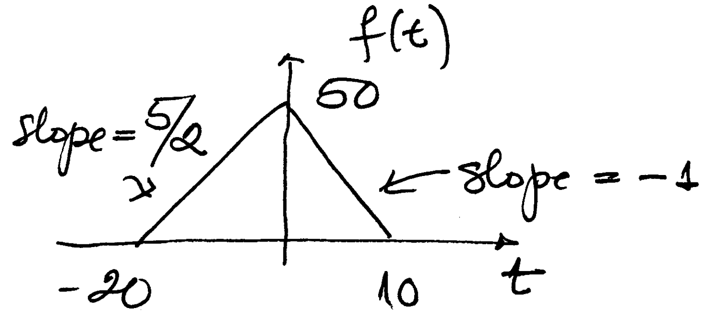
(a)



$$f(t) = \frac{5}{2}\cdot(t+20)\left[u(t+20) - u(t)\right]$$
$$- 5(t-10)\left[u(t) - u(t-10)\right]$$

$$f(t) = \frac{5}{2}\cdot(t+20)u(t+20) - \frac{5}{2}\cdot(t+20)u(t)$$
$$- 5(t-10)u(t) + 5\cdot(t-10)u(t-10)$$

$$f(t) = \frac{5}{2}(t+20)u(t+20) - \frac{5}{2}t\,u(t) - \cancel{50u(t)}$$
$$- 5t\,u(t) + \cancel{50u(t)} + 5(t-10)u(t-10)$$

$$f(t) = \frac{5}{2}\cdot(t+20)u(t+20) - \frac{15}{2}t\,u(t) + 5(t-10)u(t-10)$$

Laplace transform:

$$F(s) = \frac{5}{2}\times\frac{1}{s^2}\times e^{20s} - \frac{15}{2}\cdot\frac{1}{s^2} + \frac{5}{s^2}e^{-10s}$$

(b) $F(s) = \dfrac{4s^2+7s+1}{s(s+1)^2}$

Partial fractions expansion leads to:

$$F(s) = \frac{K_0}{s} + \frac{K_1}{(s+1)^2} + \frac{K_2}{s+1}$$

Calculati of residus:

→ $K_0 = \lim\limits_{s\to 0} s\cdot F(s)$

$$= \lim_{s\to 0} \frac{4s^2+7s+1}{(s+1)^2}$$

$K_0 = 1$

→ $K_1 = \lim\limits_{s\to -1} (s+1)^2 \cdot F(s)$

$$= \lim_{s\to -1} \frac{4s^2+7s+1}{s}$$

$$= \frac{4\times(-1)^2+7(-1)+1}{-1}$$

$K_1 = 2$

$$\rightarrow k_2 = \lim_{s \to -1} \frac{d}{ds} \cdot (s+1)^2 \cdot F(s)$$

$$= \lim_{s \to -1} \frac{d}{ds} \cdot \frac{4s^2+7s+1}{s}$$

$$= \lim_{s \to -1} \frac{(8s+7)s - (4s^2+7s+1)}{1}$$

$$= \lim_{s \to -1} (4s^2-1)$$

$$k_2 = 3$$

Hence:

$$F(s) = \frac{1}{s} + \frac{2}{(s+1)^2} + \frac{3}{s+1}$$

inverse Laplace transform:

$$f(t) = u(t) + 2t e^{-t} u(t) + 3e^{-t} u(t)$$

or

$$f(t) = (1 + 2te^{-t} + 3e^{-t}) u(t)$$

Initial-value theorem:

$$\lim_{s\to\infty} s\cdot F(s) = \lim_{t\to 0^+} f(t)$$

$$\lim_{s\to\infty} s\cdot F(s) = \lim_{s\to\infty} \frac{4s^2+7s+1}{(s+1)^2}$$

$$= 4$$

Note that $f(0) = 4$ ✓

Final Value Theorem:

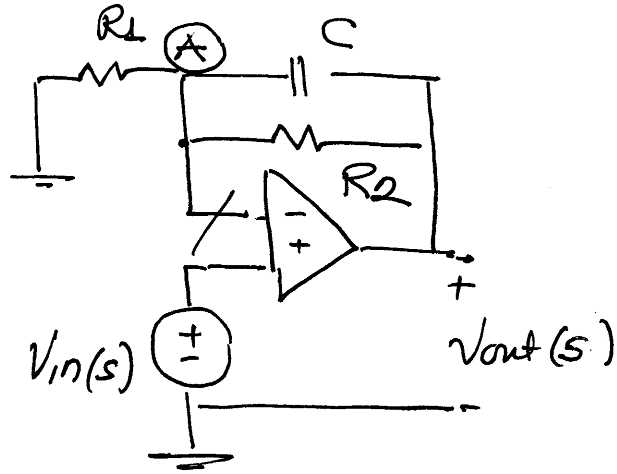
$$\lim_{s\to 0} s\cdot F(s) = \lim_{t\to\infty} f(t)$$

$$\lim_{s\to 0} s\cdot F(s) = \lim_{s\to 0} \frac{4s^2+7s+1}{(s+1)^2}$$

$$= 1$$

Again $f(\infty) = 1$ ✓

#3.



$$V_A = V_{in}(s)$$

Nodal equation in "s" domain:

$$\frac{V_{in}(s)}{R_1} + \frac{V_{in}(s) - V_{out}(s)}{R_2} + \frac{V_{in}(s) - V_{out}(s)}{1/Cs} = 0$$

Hence:

$$V_{in}(s) \cdot \left(1/R_1 + 1/R_2 + Cs \right) = V_{out}(s) \cdot \left(1/R_2 + Cs \right)$$

Transfer function is:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)}$$

$$H(s) = \frac{1/R_1 + 1/R_2 + Cs}{1/R_2 + Cs}$$

$$H(s) = \frac{s + \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}{s + \frac{1}{CR_2}}$$

If $H(s) = \frac{s+4}{s+2}$

By comparison: $\frac{1}{C}\cdot\left(\frac{1}{R_1}+\frac{1}{R_2}\right) = 4$ (1)

$\frac{1}{CR_2} = 2$ (2)

Since $C = 1\mu F$ $R_2 = \frac{1}{2C}$ From (2)

$R_2 = 500\ k\Omega$

From (1): $\frac{1}{CR_1} = 2$; $R_1 = 500\ k\Omega$

Step respons of the circuit:

$H(s) = \frac{s+4}{s+2}$

$V_{out}(t) = \mathcal{L}^{-1} H(s)\cdot V_{in}(s)$

$V_{in}(s) = 1/s$ (Step fuct)

$V_{out}(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s(s+2)}\right\}$

Since: $\frac{s+4}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$ $A = \lim_{s\to 0} s\cdot\frac{s+4}{s(s+2)}$

$= \frac{2}{s} - \frac{1}{s+2}$ $A = 2$

$B = \lim_{s\to -2} (s+2)\cdot\frac{s+4}{s(s+2)}$

$B = -1$

$V_{out}(t) = \left(2 - e^{-2t}\right)u(t)$

#4 $H(s) = \dfrac{2s+4}{s^2+5s+6}$

Zeros and Poles:

Zeros (2): $s_1 = \infty$
$s_2 = -2$

Poles (2): $s^2+5s+6=0$ $\quad s_1 = -2$
$(s+2)(s+3)=0$ $\quad s_2 = -3$

$v_{in}(t) = 4\cos(2t + \pi/4)$ $\quad \omega = 2$ rad/s
$\varphi = \pi/4$ rad

$$H(2j) = \frac{2(2j+4)}{(2j)^2+5\times 2j+6}$$

$$= \frac{4+4j}{-4+10j+6}$$

$$= 2\cdot\frac{2+2j}{2+5j\times 2} \quad ; \quad H(2j) = \frac{2+2j}{1+5j}$$

$V_{out}(j\omega) = H(j\omega)\times V_{in}(j\omega)$ $\quad \omega = 2$

$$V_{out}(j\omega) = \frac{2+2j}{1+5j}\times 4\times e^{j\pi/4}$$

$$= \frac{2\sqrt{2}}{\sqrt{26}}\cdot\frac{e^{j\pi/4}}{e^{j\arctan 5}}\times 4\times e^{j\pi/4}$$

$$= \frac{8}{\sqrt{13}}\, e^{j(\pi/2-\arctan 5)}$$

Hence, $V_{out}(t) = \frac{8}{\sqrt{13}}\cdot\cos(2t+\pi/2-\arctan 5)$

Or, equivalently:

$$V_{out}(t) = -\frac{8}{\sqrt{13}} \cdot \sin\left(2t - \arctg 5\right)$$