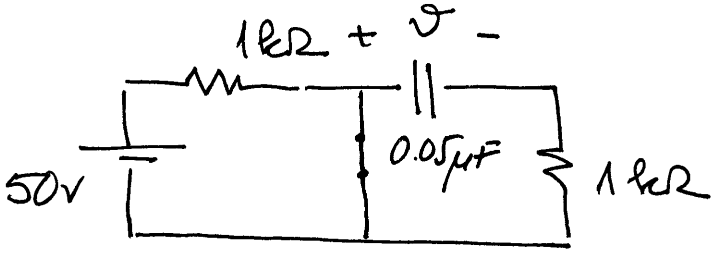
Midterm Exam #1: Summer 2006

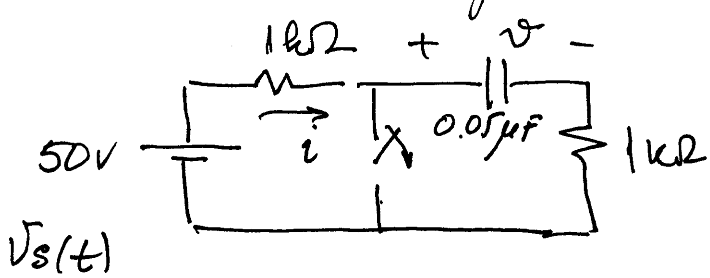
①




$t<0$: switch is closed

$v(0-) = 0$ 1st initial condition

$t=0$: switch opens




$C = 0.05\mu F$
$R = 1.000\Omega$

$$V_s(t) - 2\cdot R i - v = 0$$

$$i = C\cdot dv/dt$$

$$2RC\cdot\frac{dv}{dt} + v = v_s(t)$$

$$2\times 1\times 10^3\times 0.05\times 10^{-6}\cdot\frac{dv}{dt} + v = 50$$

$$0.1\times 10^{-3}\cdot\frac{dv}{dt} + v = 50 \qquad \text{(Differential equation)}$$

Characteristic equat:

$$0.1\times 10^{-3}\cdot s + 1 = 0 \ ; \quad s = -10^4$$

Time constant:

$$\tau_1 = 10^{-4}\,s$$

$$\tau_1 = 100\,\mu s$$

Solution to the differential equation:

$$v = k e^{-\frac{t}{10^{-4}}} + A$$

$v_p(t)$ : particular solution

$$v(0+) = 0$$

$$0 = k + A \quad ; \quad k = -A$$

$v_p(t) = A$ : constant

It should satisfy the differential equation:

$$0.1 \times 10^{-3} \cdot \frac{dv}{dt} + v = 50$$

$$0 + A = 50 \quad ; \quad A = 50$$

Hence, $k = -50$

Solution:

$$v = 50\left(1 - e^{-\frac{t}{10^{-4}}}\right)$$

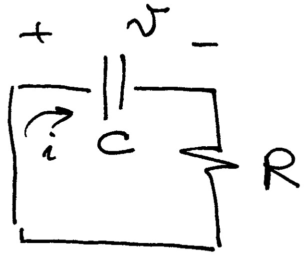
$t$: in seconds

in Volts

* If $t$ in $\mu$seconds

$$v = 50\left(1 - e^{-\frac{t}{100}}\right)$$

At $t = 100\mu s$, the switch closes




$$v = 50\left(1 - e^{-t/100}\right)$$

At $t = 100\mu$seconds

$$v(100-) = 50\cdot\left(1 - e^{-\frac{100}{100}}\right)$$

$$= 50\left(1 - e^{-1}\right)$$

$$= 50\cdot\frac{e-1}{e}$$

$$v(100+) = v(100-)$$

Hence: $v(100+) = 50\,\dfrac{e-1}{e}$

KVL: $v + Ri = 0$

$$i = C\,dv/dt$$

$$RC\cdot\frac{dv}{dt} + v = 0$$

$$1000 \times 0.05 \times 10^{-6}\cdot\frac{dv}{dt} + v = 0$$

$$50 \times 10^{-6}\cdot\frac{dv}{dt} + v = 0$$

Characters equat.: $50 \times 10^{-6}\cdot s + 1 = 0 \quad s = -\dfrac{10^{6}}{50}$

$$s = -20 \times 10^{4}$$

Time constant $\tau_2 = 50\times10^{-6}$ in seconds

$$\tau_2 = 50\,\mu sec.$$

Solution:

$$v = K\cdot e^{-\frac{t}{50\times10^{-6}}}$$

$$v(100^+) = 50\cdot\frac{e-1}{e}$$

$$50\cdot\frac{e-1}{e} = k\cdot e^{-\frac{100\times10^{-6}}{50\times10^{-6}}}$$

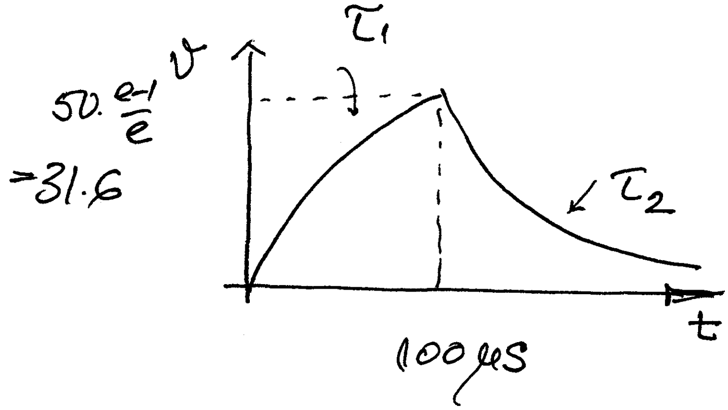
$$k = 50\cdot\frac{e-1}{e}\, e^{2}$$

$$k = 50e(e-1)$$

$v = 50e(e-1)\times e^{-t/50\times10^{-6}}$ t in seconds

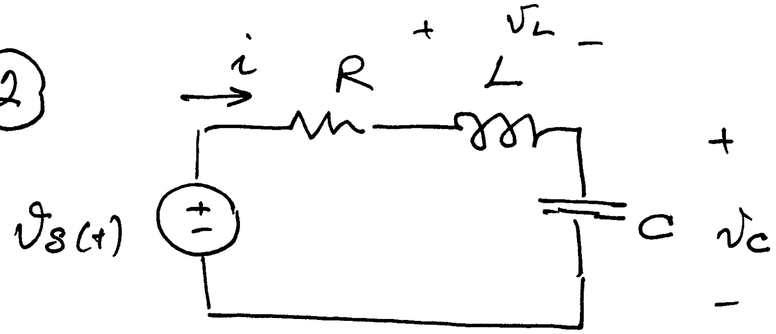
* If t in μseconds.

$$v = 50e(e-1)\cdot e^{-t/50}$$




$e = 2.718281828$

(2)




$i = i_C = i_L$

KVL: $v - Ri - v_L - v_C = 0$

↑ Ohm's Law

$$i = C \cdot \frac{dv_C}{dt}$$

$$v_L = L \cdot \frac{di_L}{dt}$$

$$R \cdot i + L \cdot \frac{di_L}{dt} + v_C = v_S$$

$$i_L = C \cdot \frac{dv_C}{dt}$$

$$\left.\begin{aligned} \frac{di_L}{dt} &= -\frac{R}{C} \cdot i_L - \frac{1}{L} \cdot v_C + \frac{1}{L} \cdot v_S(t) \\ \frac{dv_C}{dt} &= \frac{1}{C} \cdot i_L \end{aligned}\right\}$$

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} \cdot v_S(t)$$

$$\det(sI - A) = 0$$

$$\det\left( s \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \right) = 0$$

$$\det\begin{pmatrix} s+R/L & 1/L \\ -1/C & s \end{pmatrix}=0$$

$$s(s+R/L)+\frac{1}{LC}=0$$

$$s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

$$s_{1/2}=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

Case:

#1: $s_1 \neq s_2$; both natural frequencie are real

#2: $s_1 = s_2$ : $\left(\frac{R}{2L}\right)^2-\frac{1}{LC}=0$

$$\frac{R}{2L}=\frac{1}{\sqrt{LC}}$$

$$R=2\cdot\sqrt{\frac{L}{C}}$$

Natural frequencies are real and identical

#3 $s_1 \neq s_2$ : both natural frequencies are complex. (conjugate)

$$s_{1/2}=-\frac{R}{2L}\pm j\sqrt{\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}$$

Clasification:

#1: Overdamped

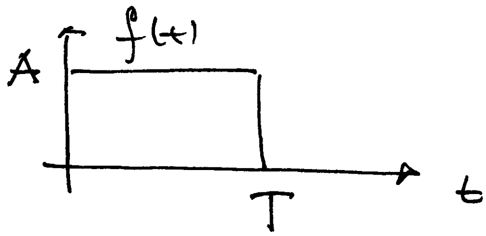
#2: Critically damped

#3: Underdamped

Special case: $R=0$: $s_{1/2}=\pm j\cdot\frac{1}{\sqrt{LC}}$ : underdamped case

3 a) 

f

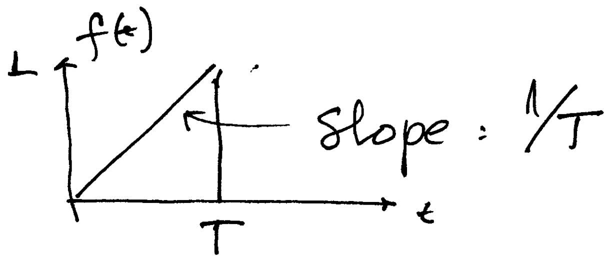



$$f(t) = A\cdot\left(u(t) - u(t-T)\right)$$

$$F(s) = A\cdot\left(1/s - \frac{1}{s}\cdot e^{-Ts}\right)$$

$$F(s) = A\cdot\frac{1-e^{-Ts}}{s}$$

b)




$$f(t) = \frac{1}{T}t\cdot[u(t) - u(t-T)]$$

$$f(t) = \frac{1}{T}\cdot t\cdot u(t) - \frac{1}{T}\cdot \underbrace{t}_{t-T+T}\cdot u(t-T)$$

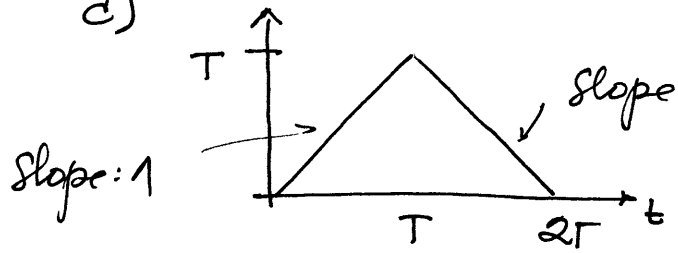
$$f(t) = \frac{1}{T}t\cdot u(t) - \frac{1}{T}\cdot(t-T+T)\cdot u(t-T)$$

$$f(t) = \frac{1}{T}\cdot t\cdot u(t) - \frac{1}{T}\cdot(t-T)\cdot u(t-T) - u(t-T)$$

$$F(s) = \frac{1}{T}\cdot\frac{1}{s^2} - \frac{1}{T}\cdot\frac{1}{s^2}\cdot e^{-Ts} - \frac{1}{s}\cdot e^{-Ts}$$

$$F(s) = \frac{1-e^{-Ts}}{Ts^2} - \frac{e^{-Ts}}{s}$$

c)



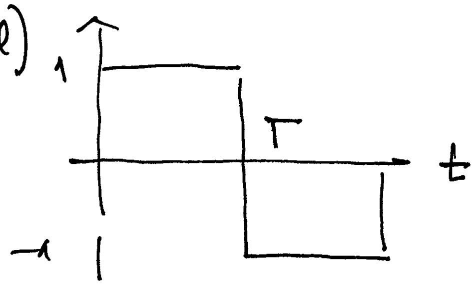

$$f(t) = t\cdot[u(t) - u(t-T)) - (t-2T)\cdot[u(t-T) - u(t-2T)]]$$

$$f(t) = t\cdot u(t) - t\cdot u(t-T) - (t-2T)u(t-T) +$$
$$(t-2T)\cdot u(t-2T)$$

$$f(t) = t\cdot u(t) - 2(t-T)\cdot u(t-T) + (t-2T)\cdot u(t-2T)$$

$$F(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-Ts} + \frac{1}{s^2}e^{-2Ts}$$

d)




$$f(t) = u(t) - u(t-T) - (u(t-T) - u(t-2T))$$

$$f(t) = u(t) - 2u(t-T) + u(t-2T)$$

$$F(s) = \frac{1}{s} - \frac{2}{s}e^{-Ts} + \frac{1}{s}e^{-2Ts}$$

(4)



* $F(s) = \dfrac{s^2+2s+2}{s(s+2)^2}$

$F(s) = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$ partial fraction expansion

$A = \lim\limits_{s \to 0} s \cdot F(s)$ ; $A = 1/2$

$B = \lim\limits_{s \to -2} \dfrac{d}{ds} (s+2)^2 F(s)$

$= \lim\limits_{s \to -2} \dfrac{d}{ds} \cdot \dfrac{s^2+2s+2}{s}$

$= \lim\limits_{s \to -2} \dfrac{d}{ds} \left(s + 2 + \dfrac{2}{s}\right)$

$= \lim\limits_{s \to -2} \left(1 - 2/s^2\right)$

$= 1 - 2/4$

$= 1/2$

$C = \lim\limits_{s \to -2} (s+2)^2 \cdot F(s)$

$= \lim\limits_{s \to -2} \dfrac{s^2+2s+2}{s}$

$= \dfrac{4-4+2}{-2}$ ; $C = -1$

Hence,

$$F(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{(s+2)^2}$$

$$f(t) = \frac{1}{2} \cdot u(t) + \frac{1}{2} e^{-2t} u(t) - t \cdot e^{-2t} u(t)$$

$$f(t) = \left( \frac{1}{2} + \frac{1}{2} e^{-2t} - t e^{-2t} \right) u(t)$$

* $$F(s) = \frac{2s e^{-2s}}{(s+1)(s+3)}$$

$$\hat{F}(s) = \frac{2s}{(s+1)(s+3)}$$

Partial fracti expansn.

$$\hat{F}(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = \lim_{s \to -1} (s+1) \cdot \hat{F}(s)$$

$$= \lim_{s \to -1} \frac{2s}{s+3} \; ; \; A = -1$$

$$B = \lim_{s \to -3} (s+3) \cdot \hat{F}(s)$$

$$= \lim_{s \to -3} \frac{2s}{s+1} \; ; \; B = 3$$

$$\hat{F}(s) = \left( -\frac{1}{s+1} + \frac{3}{s+3} \right) \; ; \; F(s) = \left( -\frac{1}{s+1} + \frac{3}{s+3} \right) e^{-2s}$$

$$\hat{f}(t) = \left( -e^{-t} + 3e^{-3t} \right) u(t)$$

$$f(t) = \left( -e^{-(t-2)} + 3e^{-3(t-2)} \right) u(t-2)$$