ENSC 320 Electric Circuits II (Summer 2006)

This is the second course in electric circuits. ENSC 220 Electric Circuits I is its prerequisite. See any accessible web pages for ENSC 220 (http://www.ensc.sfu.ca/~ljilja/ENSC220/). (Refresher: Notes, PDF: http://www.ensc.sfu.ca/people/faculty/ljilja/ENSC220/ensc220a.pdf.)

The goal of the course is to introduce formal tools and methods necessary to analyze various physical systems (such as circuits). These tools are general and may be applied to other engineering systems: control systems, mechanical systems, MEMS.

In this course we will apply these formal methods and tools to analyze various circuits. In particular, we will analyze linear circuits. They can be of various complexity:

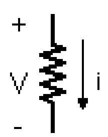
- first order
- second order
- complex circuits (op-amps, ICs).

Model: The first step in analyzing a physical engineering system is to model it. Models of physical systems may be:

- linear
- non-linear.

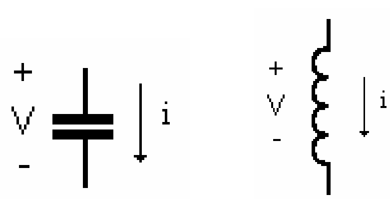
All physical systems are non-linear. We approximate their behaviour with linear models because they are easier to analyze.

Examples:
Resistor (characterized by resistor R):



R: constant (Ohm's Law: $v = Ri$)
R: may be a function of v and/or i
$R = f(v, i)$ {nonlinear function}
R: may be a function of time
$R = f(t)$ {time-varying resistor (varistor)}

Similarly, for capacitors and inductors:

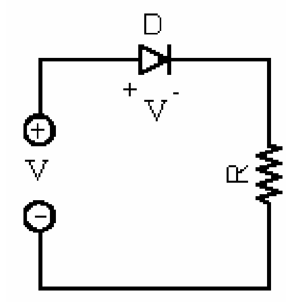


A simple, nonlinear circuit:

D: diode

v: f_1 (i)

i: f_2 (v)



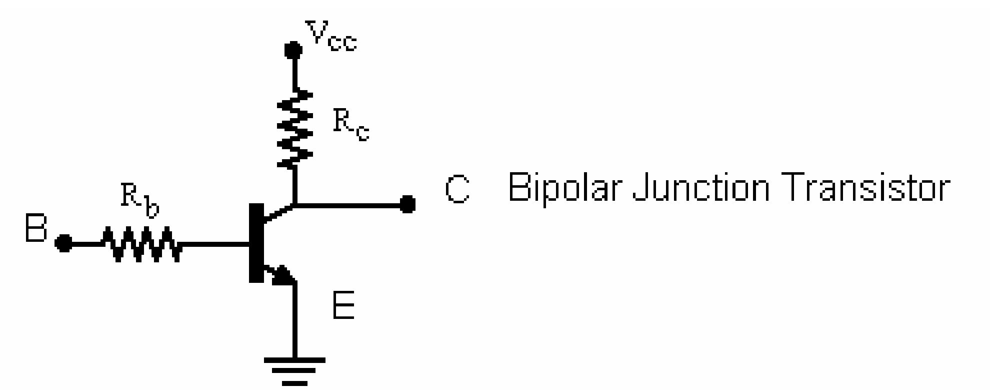
IC diode: Esaki or exponential diode $i = f(v)$

$$i = I_s e^{-nv}$$

parameters: $I_s = 10^{-10}$–10^{-14} (A)
n = ~40 (at room temperature)

This one nonlinear element makes the entire circuit a nonlinear one.

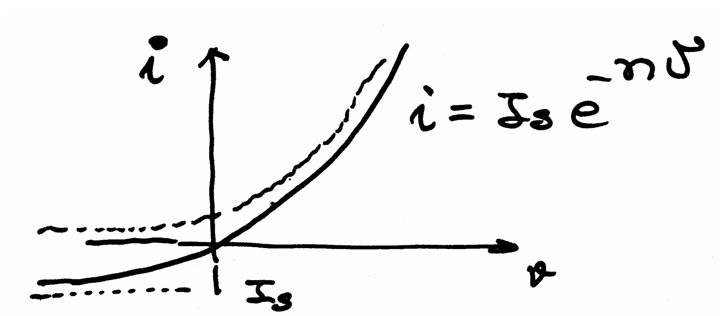
A common nonlinear circuit is a circuit that consists of transistors and resistors:



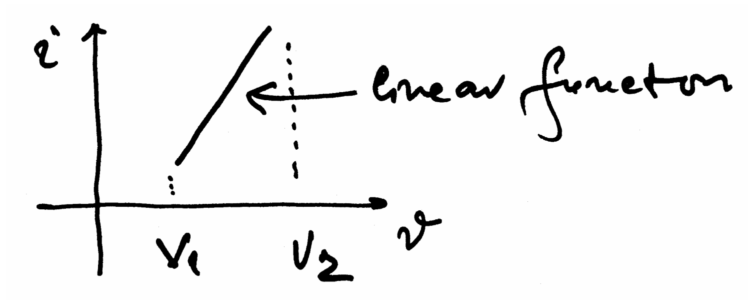
The BJT is a nonlinear element.

Behaviour of nonlinear elements (and circuits) may be approximated by linear elements that capture the essential behaviour of interest. Nevertheless, they are only approximations that are valid over a limited range of variables (voltages, currents, temperature, time).

Example: nonlinear diode



May be approximated over a certain (limited) range of voltage as a linear resistor.



This linear approximation is valid only over a limited range of V, such that $V_1 < v < V_2$. We will assume that the circuit models under analysis are linear. This will enable us to apply mathematical tools such as:

- differential calculus: equations
- transforms: Laplace, Z, Fourier, wavelets

We will be able to analyze circuits in:

- time domain
- frequency domain

Recall ENSC 380: Linear Systems.

Similar approach, applied to general engineering systems (control systems).

Linear systems have "nice" properties and the equations emanating from their descriptions have analytical (closed form) solutions. Linear systems may be described with ordinary differential equations that have closed form solutions. The degree of differential equations depends on the complexity of the system.

We will study 1st and 2nd order circuits. In the time domain (reality), they are described with 1st and 2nd order ODEs (ordinary differential equations).

Note: transmission lines may be modeled as ladder structures of infinite size. They are "distributed" linear circuits and are described by partial differential equations (PDEs).

Linear system:

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

A system is linear if and only if the response to $x_1(t) + x_2(t) = y_1(t) + y_2(t)$, where

$$x_1(t) \longrightarrow y_1(t)$$
$$x_2(t) \longrightarrow y_2(t)$$

for any choice of $x_1(t)$ and $x_2(t)$.

Proposition: if $x(t) \rightarrow y(t)$
$\alpha x(t) \rightarrow \alpha y(t)$

More generally:

$$\alpha x_1(t) + \beta x_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$$ {principle of superposition}

First order Circuits (RC, RL) – *Text Ch8, Notes.pdf p120*

- circuits with R and (L or C)

Supplies:
- arbitrary function generator
- DC (direct current)
- AC (alternating current)
- other waveforms

Function generator can produce:

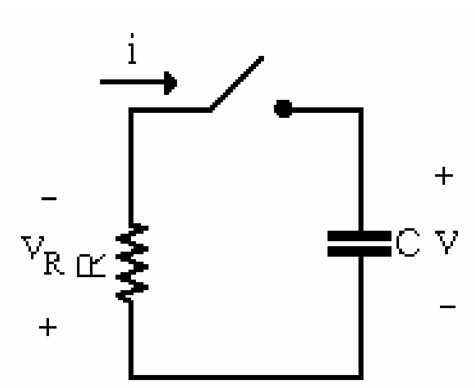
- sinusoidal
- square
- sawtooth
- triangular

all are periodic

- step
- square pulse
- ramp
- exponential

all are aperiodic

RC circuit:

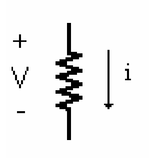


R, C : constants

linear elements --> linear circuit --> linear ODE with constant coefficients

Ohm's Law: $V_R = iR$ (associated direction)

$$i = C\frac{dV}{dt}$$



$$\left.\begin{array}{ll}\text{KCL:} & i_R = i_C \\ \text{KVL:} & V_R + V = 0\end{array}\right\}\quad \text{valid for non-linear circuits also}$$

Hence: Z constitutive relationships: $V_R = iR$

$$i = C\frac{dV}{dt}$$

Then, $R \cdot C\dfrac{dV}{dt} + V = 0$ Differential equation

$\dfrac{dV}{dt} + \dfrac{1}{RC}V = 0$ Homogeneous (linear) ODE

Solving the equation,

$$\int_{Vo}^{V(t)} \frac{dV}{V} + \int_0^t \frac{dt}{RC} = 0$$

$$\ln V \Big|_{Vo}^{V(t)} = -\frac{1}{RC} \cdot t$$

$$\ln V(t) - \ln V_0 = \frac{-t}{RC}$$

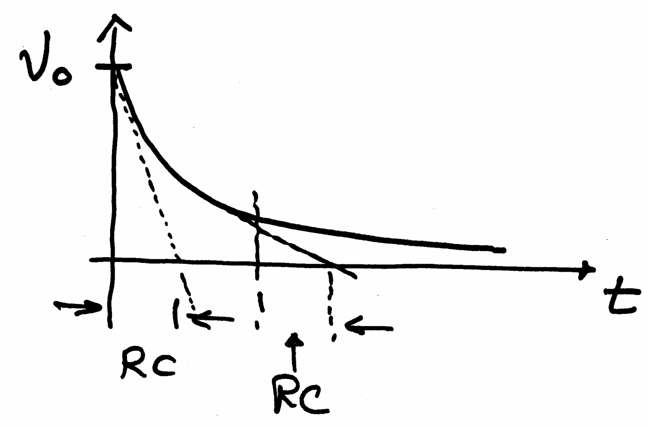
$$\ln \frac{V(t)}{V_0} = \frac{-t}{RC}$$

$$V(t) = V_0\, e^{-t/RC}$$

Hence, at t = 0 and $V(0) = V_0$

RC: time constant (dimension = time)

$\dfrac{1}{RC}$: characteristic frequency (natural frequency)



There is another approach to finding the solution to our equation:

$$\frac{dv}{dt}+\frac{1}{RC}V=0$$

Suppose that the solution is of the form

$V(t) = ke^{st}$ where k and s are unknown constants.

Let us see if we can find these constants by satisfying the equation we are trying to solve.

$$V(t) = ke^{st}$$

$$\frac{dV(t)}{dt}=kse^{st}$$

Hence, $$\frac{dV}{dt}+\frac{1}{RC}\cdot V=0$$

becomes $$kse^{st}+\frac{1}{RC}ke^{st}=0$$

$$ks+\frac{1}{RC}\cdot k=0$$

$$k\left(s+\frac{1}{RC}\right)=0\Rightarrow s=-\frac{1}{RC}$$ {s = –1/RC: natural frequency of the circuit}

Thus, $V(t) = V_0\, e^{-t/RC}$

How do we find k?

It depends on the "initial conditions". Recall from the previous methods that $V(0) = V_0$. Hence, if we know $V(t)$ for some $t = t_0$, we can find k. If $V(0) = V_0$ (as in the previous case),

$V(0) = ke^{-\frac{0}{RC}}$ ==> $V(0)$ = k and k = V_0

And, as in the previous approach:

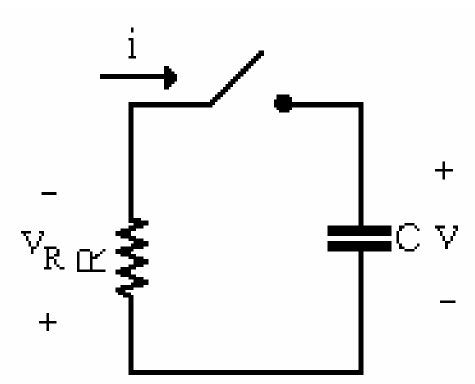
$V(t) = V_0\ e^{-t/RC}$ as expected

Observations:

Circuit: - linear with constant parameters R and C
- no input
- initially stored energy in the capacitor

Differential Equation:
- linear with constant coefficients
- homogeneous
- non-trivial ($\neq 0$) solution

Let us revisit our simple RC circuit again and make a differential equation in terms of current rather than voltage:



Again: V_R = iR

$$i = C\frac{dV}{dt}$$

$$V_R + V = 0$$

iR + V = 0 ==> V = -iR

Hence: $$i = C \cdot (-R \cdot \frac{di}{dt})$$

$$\frac{di}{dt} + \frac{1}{RC} i = 0 \qquad \text{(compare to } \frac{dV}{dt} + \frac{1}{RC} \cdot V = 0\text{)}$$

Both equations have the same format of a solution:

$y = Ae^{-\frac{t}{RC}}$ where A depends on the variable:

V_R

V_C <-- V

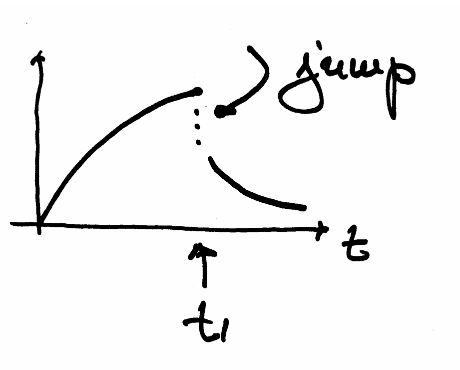
$\left.\begin{array}{l} i_R \\ i_C \end{array}\right\}$ i

If we know the value of y at some instant t_1, we can find A:

$$y(t_1) = Ae^{-\frac{t_1}{RC}}$$

Hence, $A = y(t_1)e^{\frac{t_1}{RC}}$

Shape of $\left.\begin{array}{l} i_C \\ i_R \end{array}\right\}$



Since V_R = iR (where V_R can also have a "jump")

Nevertheless, V(t) being the voltage across the capacitor has to be a continuous function of time.

Why? V_C(t) cannot have "jumps"?

Since, $i_C = C \cdot \frac{dV_C}{dt}$ if there is a discontinuity then $i_C \to \infty$. This is impossible in a model that mimics a physical system.

Summary:

- write differential equations in terms of V_C(t)
- use one known value of V_C(t) at one t to find the constant term in:

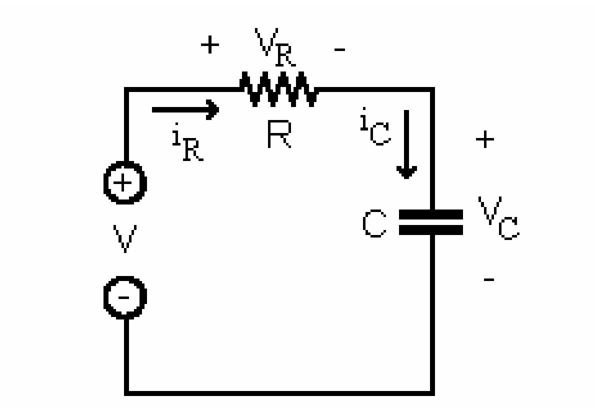
$$V_C(t) = V_C e^{-\frac{t}{RC}}$$

Usually, V_C(0) is known (though not necessarily the only instance).

V_C(0-) = V_C(0+)

continually holds instance of closing or opening the switch

The RC circuit we have analyzed is the simplest case. A bit more complicated circuit is an RC circuit with input.



$$\left.\begin{aligned} V_R &= R \cdot i_R \\ i_C &= C \cdot \frac{dV_C}{dt} \end{aligned}\right\} \text{constitutive relationship}$$

$$i_R - i_C = 0 \qquad \text{<-- KCL}$$

$$V_S - V_R - V_C = 0 \qquad \text{<-- KVL}$$

Hence,

$$R \cdot i_R + V = V_S$$

$$i_R = i_C$$

$$\rightarrow R \cdot i_C + V_C = V_S$$

$$R \cdot C \frac{dV_C}{dt} + V_C = V_S$$

driving force V_S(t): function of time

$$\underbrace{\frac{dV_C}{dt} + \frac{1}{RC} \cdot V_C}_{\text{the same as in the case of the RC circuit without input}} = \frac{1}{RC} \cdot V_S$$

This is a differential equation:

- linear
- first order
- constant coefficients
- non-homogeneous

Homogeneous part:

$$\frac{dV_C}{dt} + \frac{1}{RC} \cdot V_C = 0$$

Method for finding a solution to the complete equation:

Suppose that we can express the complete solution as:

$$V_C(t) = V_h(t) + V_p(t)$$

particular solution that depends on $V_s(t)$

satisfies the entire equation

homogeneous solution satisfies only the homogeneous part of the ODE

Proof: substitute $V_C(t)$ in the equation keeping in mind that

$V_h(t)$ satisfies the homogeneous part
$V_p(t)$ satisfies the complete equation

Solution to:

$$\frac{dV}{dt} + \frac{1}{RC} V = 0$$

is already known to us (from previous example of RC circuit without a source):

$$V_h = ke^{-\frac{t}{RC}}$$

What is $V_p(t)$?

It's form depends on $V_S(t)$ – the right-hand side of the differential equation.

It is called a particular solution.

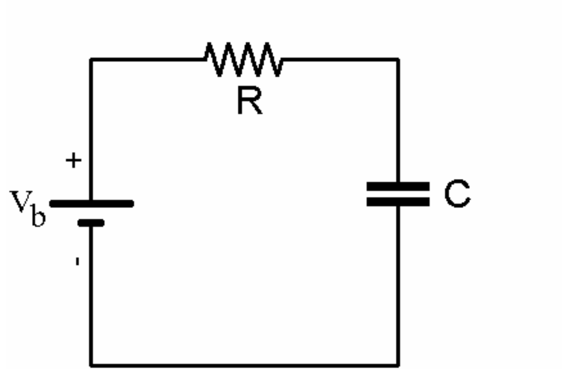
$V_p(t)$ (particular because it depends on the type of driving function $V_s(t)$)

Case 1:

$V_s(t) = V_b$ (battery (DC source))

V_b = constant (5V, 10V)

Circuit:



Since $V_s(t)$ is a constant, it is natural to suspect that Vp(t) may be also a constant. Hence, let us assume that

Vp(t) = Vp (constant: unknown)

Since Vp(t) has to satisfy the entire differential equation:

$$\frac{dV_p}{dt} + \frac{1}{RC}V_p = \frac{V_b}{RC}$$

V_p is a constant, hence $\frac{dV_p}{dt} \equiv 0$

leads to

$V_p = V_b$ is a solution.

The complete solution to the DE is:

$Vc(t) = Vb + ke^{-\frac{t}{RC}}$

k is still unknown
V_b is a constant (because $V_s(t)$ is a constant DC source)
constant k depends on the initial conditions. Usually, $V_c(0)$ is known; if capacitor C was not charged:

$V_c(0-) = 0$ and $V_c(0+) = 0$ ($V_c(t)$ is a continuous function of time)

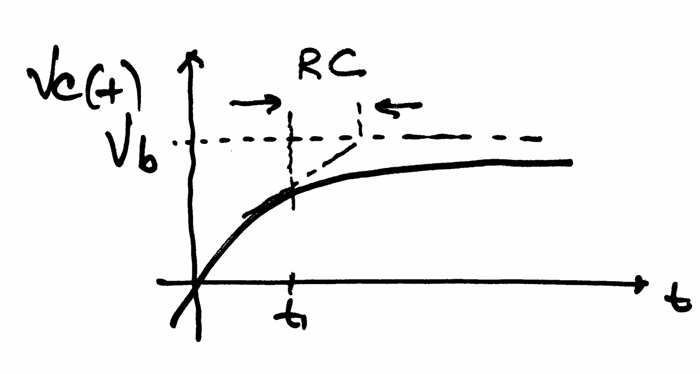
Hence, let $V_c(0-) = 0$, then

$$V_c(0+) = V_b + ke^{-\frac{0}{RC}}$$
$$V_c(0+) = V_b + k \Rightarrow k = -V_b$$

Our solution is:

$$V_c(t) = V_b(1 - e^{-\frac{t}{RC}})$$

which is the complete solution to our ODE.

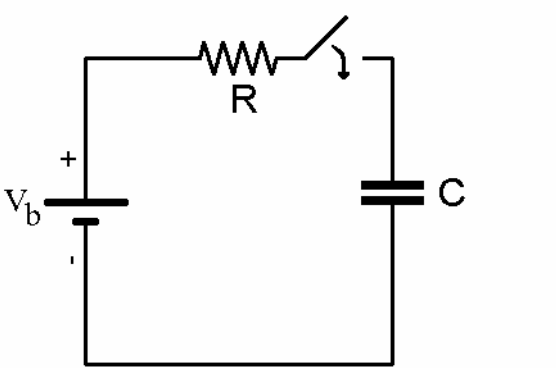


(again: prove that we can find RC by drawing a tangent anywhere)

At time: $t=0$, $V_c(0+) = 0$

$t \to \infty \Rightarrow V_c(\infty) \to V_b$ {capacitor is being charged}

Note: are these two circuits equivalent?



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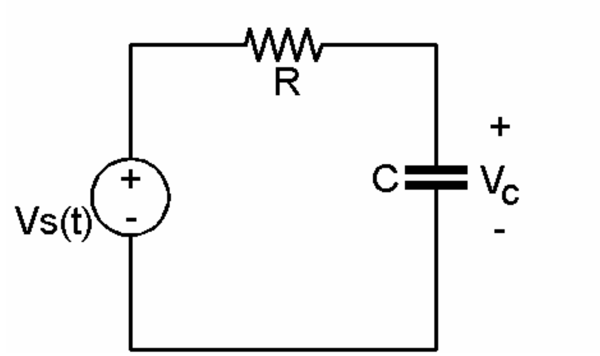

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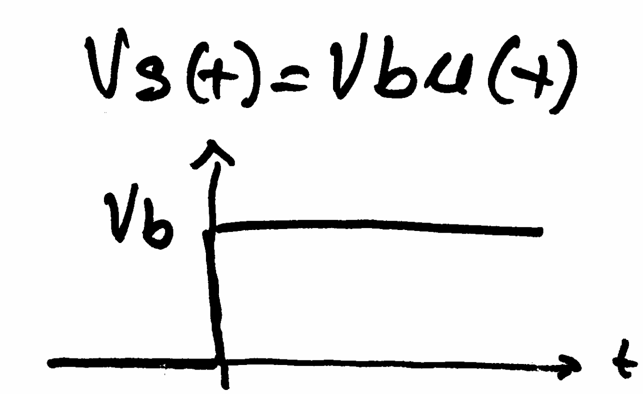
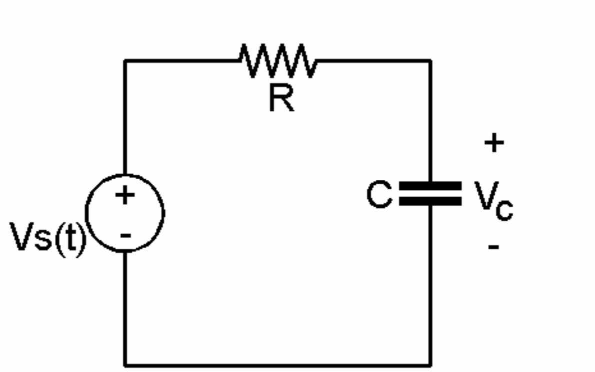
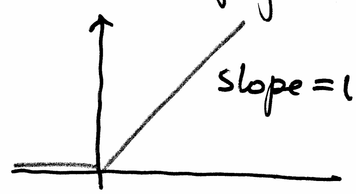


Case 2:



Vs(t): ramp function



$Vs(t) = t \cdot u(t)$

$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$

Step function

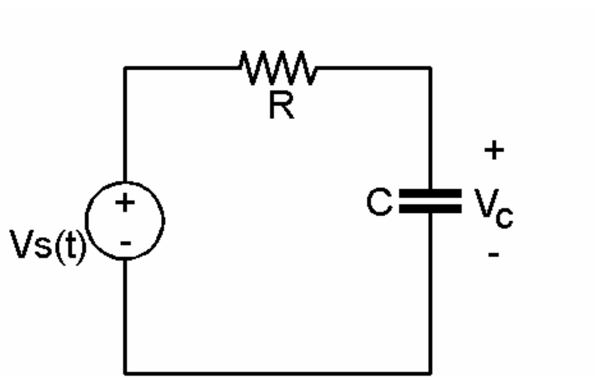
Again,

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{1}{RC}V_s(t)$$
$$V_c(t) = Vp(t) + Vh(t)$$
$$Vh(t) = ke^{-\frac{t}{RC}}$$
$$Vp(t) = (A + Bt)u(t)$$

where (A + Bt)u(t) is a more general form of t·u(t)

Unknowns are A, B and k (where k is a constant based on initial conditions). This is important in order to arrive at the most general (family) solution.

Case 3:

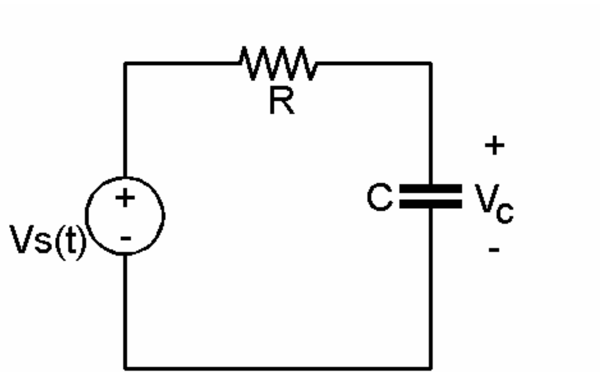


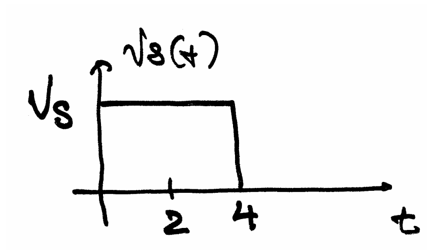
$V_s(t) = V_s e^{st} \cdot u(t)$ (u(t) ensures that $vs(t) \equiv 0$ for t < 0)

$$\frac{dV_c}{dt} + \frac{1}{RC}V_c = \frac{1}{RC}V_s(t)$$
$$Vc(t) = Vp + ke^{-\frac{t}{RC}}$$

$Vp(t) = Ae^{st}$ (what if $s = \frac{1}{RC}$?, a special case to consider)

Case 4:





Vs = 15V
R = 200
C = 10 uF

(see pp120-154 of the ENSC220 Notes.pdf)

A similar approach applies to solving higher order ODEs, emanating from higher order circuit models *(see pp155-244, ENSC220 Notes.pdf)*

General approach		Text	
Circuits are:	- linear	Ch.8	1^{st} order
	- lumped (not distributed)	Ch.9	2^{nd} order
	- constant coefficients		circuits

They are described by linear, ordinary DEs with constant coefficients.

What should be the variables chosen to appear in the final differential equation(s)?

State Variables:

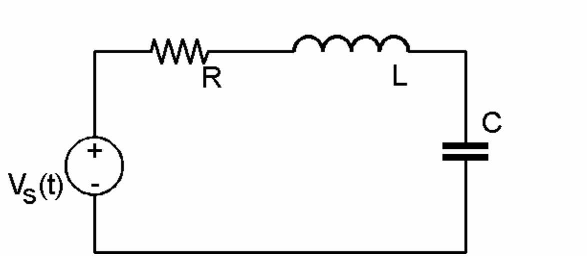
all other variables can be then expressed via:

- constitutive relationship

- KCL/KVL

- simple integration or differentiation stemming from the constitutive relationship for circuit elements

A systematic way of writing equations:



KCL: $i_R = i_L = i_C$

$$Ri_L + L\frac{di_L}{dt} + V_c = V_s\ (t)$$
$$i_L = C\frac{dV_c}{dt}$$

DE:

$$\frac{di_L}{dt} = -\frac{R}{L}i_L - \frac{1}{L}V_c + \frac{1}{L}V_s(t)$$
$$\frac{dV_c}{dt} = \frac{1}{C}i_L$$

Matrix form:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ V_c \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}\begin{pmatrix} i_L \\ Vc \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix}V_s(t)$$

where i_L = current through L
V_c = voltage across C
and both i_L and V_c are state variables

$$\frac{dx}{dt} = Ax + b$$
$$A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$$
$$b = \begin{pmatrix} 1/L \\ 0 \end{pmatrix}V_s(t)$$

A: depends on circuit topology and circuit elements and their values

b: depends on circuit topology, elements and their values and the type of sources connected to the circuit.

Complete equation (matrix form):

$$\frac{dx}{dt} = Ax + b$$

Homogeneous equation

$$\frac{dx}{dt} = Ax$$

Suppose that the solution is of the form:

$$x = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{st} \qquad\qquad \frac{dx}{dt} = s \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{st}$$

Substitution into the homogeneous equation leads to

$$s \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{st} = A \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{st}$$

$$(s \cdot I - A)\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{st} = 0$$

(Note: $e^{st} \neq 0$ and $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, ==> trivial solution)

Hence, the only other nontrivial solution ($x(t) \equiv 0$) is for:

$\det (s \cdot I - A) = 0$ Note: $(s \cdot I - A)^{-1}$ does not exist (is a singular matrix)

For the case we consider (RLC circuit)

$$A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$$

gives

$$\det\begin{pmatrix} s + R/L & 1/L \\ -1/C & s \end{pmatrix} = 0$$

or

$$s(s + \frac{R}{L}) + \frac{1}{LC} = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \quad \text{characteristic equation.}$$

Solutions of this equation with natural frequency of the circuit:

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{(\frac{R}{2L})^2 - \frac{1}{LC}}$$

$$s_{1/2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$s_{1/2} = -\alpha \pm j\omega \qquad \alpha = \frac{R}{2L} \qquad j\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Case 1: s_1 and s_2 are real and distinct

$$V_{c_h}(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} \qquad s_{1/2}\text{: real, negative}$$

Case 2: $s_1 = s_2$

$$s_1 = -\alpha \qquad \alpha = \frac{R}{2L}$$

$$V_{c_h}(t) = k_1 e^{-\alpha t} + k_2 t e^{-\alpha t}$$

Note: the most general form has to be considered. Form $k_1e^{-\alpha t} + k_2e^{-\alpha t}$ is not general and it simplifies to $ke^{-\alpha t}$.

Case 3: $s_{1/2} = -\alpha \pm j\omega$

$$V_{c_h}(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$$
$$V_{c_h}(t) = e^{-\alpha t}(k_1 e^{j\omega t} + k_2 e^{-j\omega t})$$
$$V_{c_h}(t) = e^{-\alpha t}\{(k_1 + k_2)\cos\omega t + j(k_1 - k_2)\sin\omega t\}$$
$$V_{c_h}(t) = e^{-\alpha t}(A\cos\omega t + B\sin\omega t)$$
$$V_{c_h}(t) = e^{-\alpha t} V\sin(\omega t + \theta)$$

Unknown constants: k_1 A V
k_2 B θ

equivalent representations

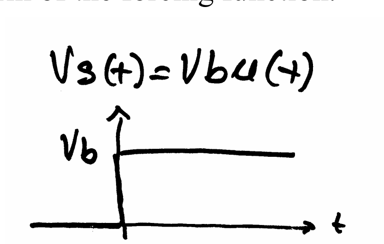
They are to be found from the initial conditions:

$V_C(0-) \equiv V_C(0+)$

$i_L(0-) \equiv i_L(0+)$ <-- substitute t = 0 in the expressions

We also need particular solutions that depend on the form of the forcing function.

$V_S(t) = V_b u(t)$



$$\begin{pmatrix} i_L p(t) \\ V_C p(t) \end{pmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

where

$$\begin{pmatrix} i_L p(t) \\ V_C p(t) \end{pmatrix}$$ satisfy the entire DE equation

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$ are unknowns

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} V_b$$ *{general procedure: Text Chapter 9}*