Recap:
We considered the two-dimensional case: an RLC circuit with two state variables.

State Equations:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} v_S(t)$$

$$\frac{dx}{dt} = Ax + b \leftarrow \text{canonical form}$$

Case 1: $s_1 + s_2$, real and negative.
Case 2: $s_1 = s_2$
Case 3: $s_1 \neq s_2$, complex conjugate

Initial condition: $v_c(0^-) = v_c(0^+); v_c(0^+) = 0$
$i_L(0^-) = i_L(0^+); i_L(0^+) = 0$

Hence:
Case 1: $v_C(t) = v_{ch}(t) + v_{cp}(t)$
Where $v_{ch}(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$ satisfies the differential equation and v_{cp} depends on the exciting function.

Canonical form of the state equation:

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} v_s(t)$$

$v_s(t) = V_b \cdot u(t)$, where $u(t)$ is the unit step function.

Assume that $i_{Lp}(t)$ and $v_{Cp}(t)$ are also constants.

Hence:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} V_b$$

Note: We use A and B as constants here to differentiate them from K_1 and K_2 in $v_{ch}(t)$.

$$-\frac{R}{L}A - \frac{1}{L}B + \frac{1}{L}V_b = 0$$

$$\frac{1}{C}A = 0 \Rightarrow A = 0$$

$$B = V_b$$

Hence:

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + V_b$$

$$\frac{dv_C(t)}{dt} = K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t} + V_b + \phi$$

Can we find $\frac{dv_C(t)}{dt}$ at $t = 0$?

This value can be found from the DE equation

$$\frac{d}{dt}\begin{pmatrix} i_L \\ v_C \end{pmatrix}_{t=0} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix}_{t=0} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} v_s(t)_{t=0}$$

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = \frac{1}{L} i_L(0+)$$

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = 0$$

Hence,

$$v_C(t)_{t=0^+} = K_1 + K_2 + V_b$$

$$K_1 + K_2 + V_b = 0$$

$$\left.\frac{dv_C}{dt}\right|_{t=0^+} = K_1 s_1 + K_2 s_2$$

$$K_1 s_1 + K_2 s_2 = 0$$

$$\left.\begin{aligned} K_1 + K_2 &= -V_b \\ K_1 s_1 + K_2 s_2 &= 0 \end{aligned}\right\} \text{multiply the first equation by } -s_2$$

$$-K_1 s_2 - K_2 s_2 = V_b s_2$$

$$K_1 s_1 + K_2 s_2 = 0$$

$$K_1 (s_1 - s_2) = V_b s_2$$

$$K_1 = \frac{V_b s_2}{s_1 - s_2} \qquad s_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 - s_2 = 2\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$K_1 = \frac{V_b}{2\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}} \cdot \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}\right)$$

Then,

$$K_2 = -\frac{s_1}{s_2} \cdot K_1$$

and

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + V_b$$

The second state variable $i_L(t)$ can be found from the DE:

$$\frac{dv_c}{dt} = \frac{1}{C} i_L$$

$$i_L = \frac{1}{C}\frac{dv_C}{dt}$$

$$i_L = \frac{1}{C}\left(K_1 s_1 e^{s_1 t} + K_2 s_2 e^{s_2 t}\right)$$

Where K_1 and K_2 have already been found for $v_C(t)$.

Note: $v_C(0^+) = 0$

$i_L(0^+) = 0$

$\left.\frac{dv_C}{dt}\right|_{t=0} = 0$ only because $\frac{dv_C}{dt} = \frac{1}{C} i_L(t)$

and

$i_L(0+) = 0$

In other circuits, we may have:

$$\left.\frac{dv_C}{dt}\right|_{t=0} \neq 0$$

$$\left.\frac{di_L}{dt}\right|_{t=0} = -\frac{R}{L} \cdot v_L(0^+) - \frac{1}{L} \cdot v_C(0^+) + \frac{1}{L} v_C(0^+)$$

$\left.\frac{di_L}{dt}\right|_{t=0^+} = \frac{V_b}{L}$ only because of the DE (constants, parameters, excitation)

In general: $\left.\frac{dv_C}{dt}\right|_{t=0^+}$ and $\left.\frac{di_L}{dt}\right|_{t=0^+} \neq 0$ even though $v_C(0^+) = i_L(0^+) = 0$

These values depend on:

- DE: circuit topology
- circuit parameters
- excitation (sources)

- initial conditions: $v_C(0^-), i_L(0^-)$

Furthermore: $\left.\frac{dv_C}{dt}\right|_{t=0^+} \neq \left.\frac{dv_C}{dt}\right|_{t=0^-}$

$\left.\frac{di_L}{dt}\right|_{t=0^+} \neq \left.\frac{di_L}{dt}\right|_{t=0^-}$

No continuity of these function is guaranteed!

Digression:
Note: One can also solve the system of two 1st order DE's by combining them into one 2nd order DE.

In the case of the RLC circuit that we analyzed:

$$\frac{di_L}{dt} = -\frac{R}{L}\cdot i_L - \frac{1}{L}v_C + \frac{1}{L}\cdot v_s(t)$$

$$\frac{dv_C}{dt} = \frac{1}{C}\cdot i_L \quad \Rightarrow \quad i_L = C\frac{dv_C}{dt}$$

$$C\frac{d^2v_C}{dt^2} = -\frac{R}{L}\cdot C\frac{dv_C}{dt} - \frac{1}{L}\cdot v_C + \frac{1}{L}\cdot v_s(t)$$

Or: $LC\frac{d^2v_C}{dt} + RC\cdot\frac{dv_C}{dt} + v_C = v_s(t)$

2nd order DE:
Characteristic equation:

$$LCs^2 + RCs + 1 = 0$$

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Again: $v_C(t) = K_1e^{s_1t} + K_2e^{s_2t} + v_{cp}(t)$
where $K_1e^{s_1t} + K_2e^{s_2t}$ is $v_{ch}(t)$, the homogenous solution.

Let us look more closely at the 2nd order DE:

$$LC\frac{d^2v_C}{dt^2} + RC\cdot\frac{dv_C}{dt} + v_C = v_s(t)$$

$$v_s(t) = V_b u(t)$$

$$\equiv V_b \; for\, t \geq 0$$

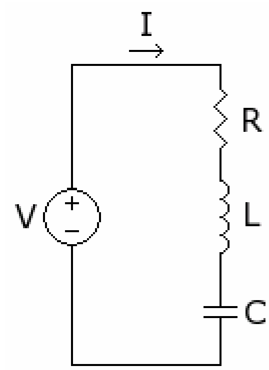
In equilibrium (steady state), we expect that:
$v_C(t) \equiv$ constant because the driving force (source function) is constant:

$v_C(t) = A$

$\frac{dv_C}{dt} = 0$; $\frac{d^2 v_C}{dt^2} = 0$ Hence: $A = V_b$

$v_C(t)_{t \to \infty} = V_b$

As expected! Look at the circuit:



$v_s(t) = V_b \leftarrow DC\ source\ for\ t \geq 0$

$v_C(t) \to V_b$

For the current flowing through the inductor

$$\frac{di_L}{dt} = -\frac{R}{L} \cdot v_L - \frac{1}{L} \cdot v_C + \frac{1}{L} \cdot v_s(t)$$

In steady state:

$$\frac{di_L}{dt} = 0 \ ; \ i_L = 0 \qquad v_C = V_b \leftarrow \text{the particular solution}$$

You can deduct many of these variables and their value at $t = 0^+$ and $t \Rightarrow \infty$ by looking at the circuit.

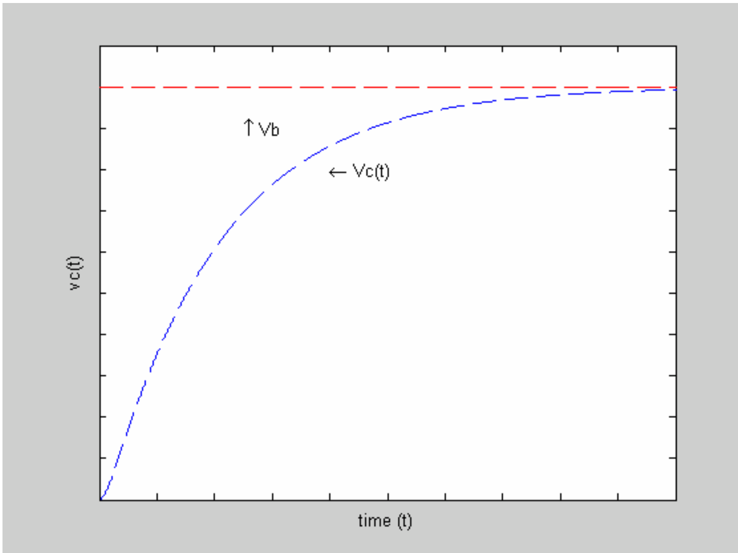
Such predictions can be done for well known source functions such as:

- DC sources: $V_b u(t) = \begin{cases} 0 & t < 0 \\ V_b & t \geq 0 \end{cases}$
- AC sources: $V_s \sin(\omega t + \theta)$

Graphical representations of the responses:

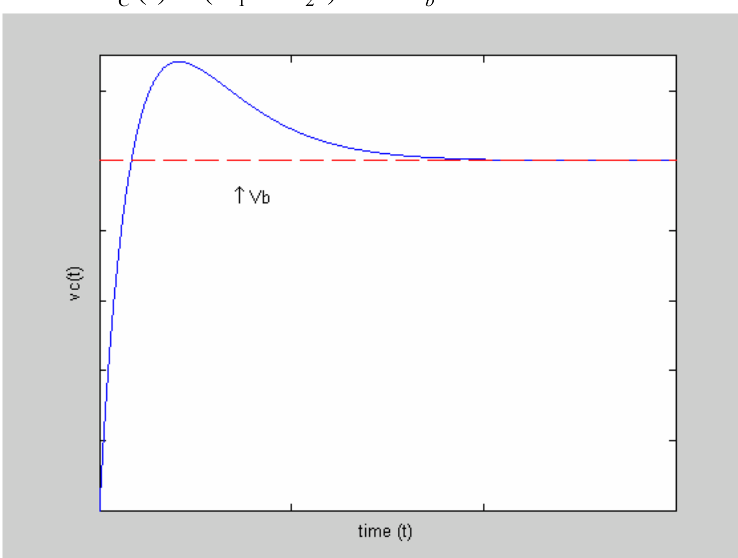
Case 1: $s_1 \neq s_2$: real numbers

$$v_C(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} + V_b$$

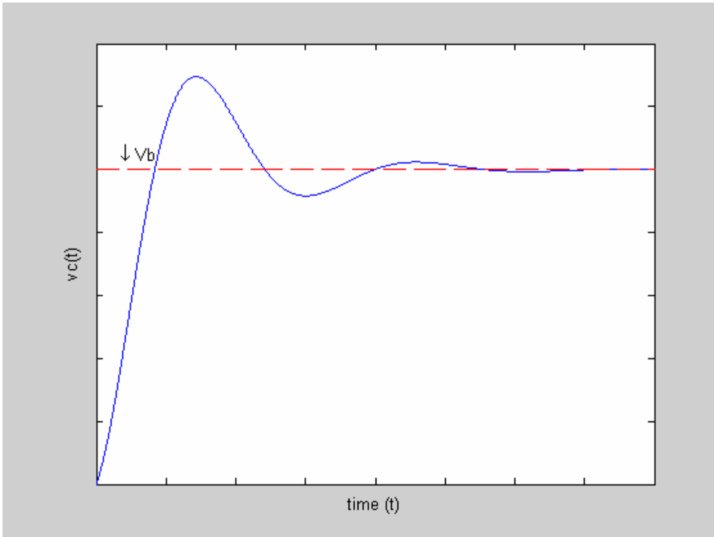


Case 2: $s_1 = s_2$: real and negative

$$v_C(t) = (K_1 + K_2 t)e^{s_1 t} + V_b$$



Case 3: $s_1 \neq s_2$: complex conjugate



Damping:

Case 1: $s_1 \neq s_2$: real and negative

$$v_{Ch}(t) = (K_1 + K_2 t)e^{s_1 t}$$

overdamped

Case 2: $s_1 = s_2$: real, negative

critically damped

It is not possible to distinguish between overdamped and critically damped responses by merely looking at the waveforms.

Case 3: $s_1 \neq s_2$: complex conjugate

damped

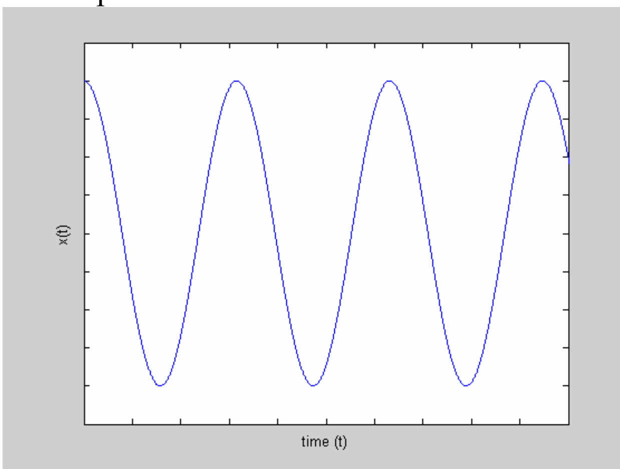
Note: if $R = 0$: undamped

$$|s_1| = |s_2|$$

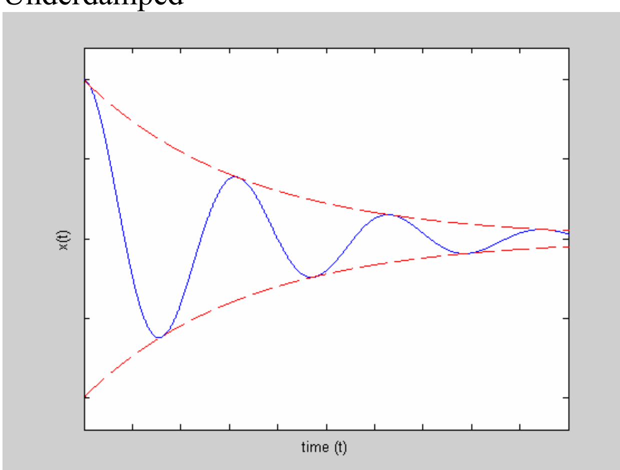
e.g., an LC circuit that sustains oscillations due to initially stored energy

Graphical representation:

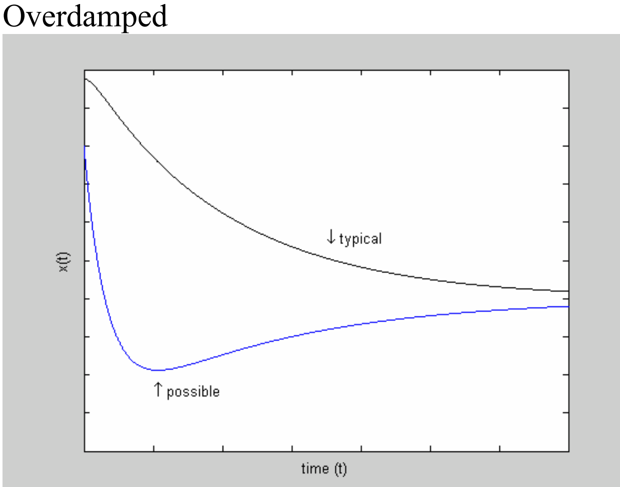
Undamped:



Underdamped



Overdamped



Critically damped