Laplace Transform (Lin & DeCarlo: Ch 13)

The Laplace transform is an integral transformation. It transforms:

$$f(t) \to F(s)$$

time variable ‘t’ complex variable ‘s’

From Euler > Lagrange > Laplace.

Hence, differential equations can be transformed to algebraic equations. Digression: the circuits we analyze are described by ODEs.

To become more mathematically correct, Circuit equations are Algebraic Differential equations

ADE

from KVL, KCL from the constitutive relationship

Laplace Transform:

ODE → algebra equations
linear → linear

or, more formally:

algebraic differential equation → algebraic equations
time domain → frequency domain
t → s

Once the system of algebraic equations is solved (there are many known methods and algorithms to do so), we have to use Inverse Laplace Transform and transform the solution

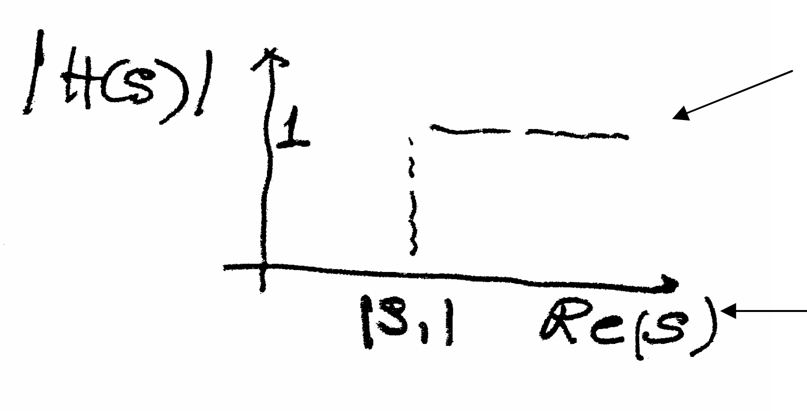
$$X(s) \to x(t)$$

capital case small case, italics ‘x’

Laplace transform greatly simplifies the process and enables us to find solutions to <u>linear</u> circuits more easily than solving differential equations in the time domain. Solving equations in the ‘s’ domain provides additional insight into the circuits behaviour.

- frequency response of the circuit: often used in the engineering approach to design circuits

Example: Filters



H(s) transfer function

frequency range

this is an ideal high pass filter

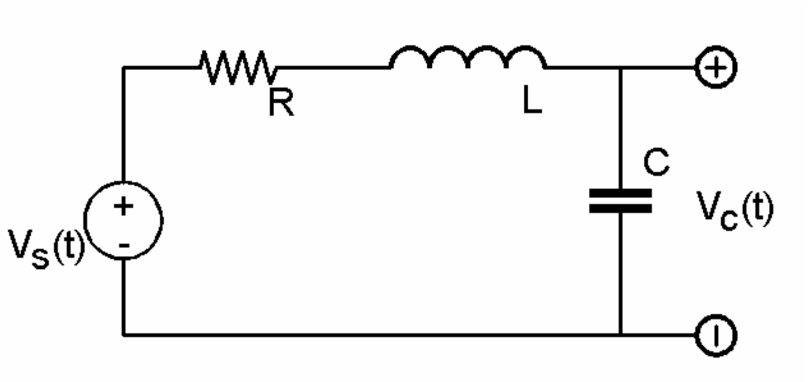
A more realistic case:

slope depends on the filter design
- Chebyslev
- elliptic

We often desire certain system performance on the frequency domain. Hence, dealing with circuit response in the 's' domain provides important insight into the system's behaviour.

What is wrong with writing and solving DEs (ODEs, ADEs)?

Procedure:



Step 1:

write DE
- Two equations of the 1st order
- One equation of the 2nd order

Step 2:

write the characteristic equation and find characteristic frequencies (natural frequencies) of the circuit

Step 3:

Predict the solution based on the forcing function and find the constants

Step 4:

Determine constants based on the circuit initial conditions and circuit equations

This works very well for simple circuits. Recall our example from last week:

1st order: RL, RC: find constant k in

$$x(t) = \underbrace{ke^{st}}_{\text{find k was 'easy'}} + \underbrace{x_p(t)}_{\text{depends on the driving function}}$$

2nd order: RLC: find k_1 and k_2

$$x(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t} + x_p(t)$$

we had to use: $x_1(0\text{-})$: V_c

$x_2(0\text{-})$: i_L

and <u>also</u>:

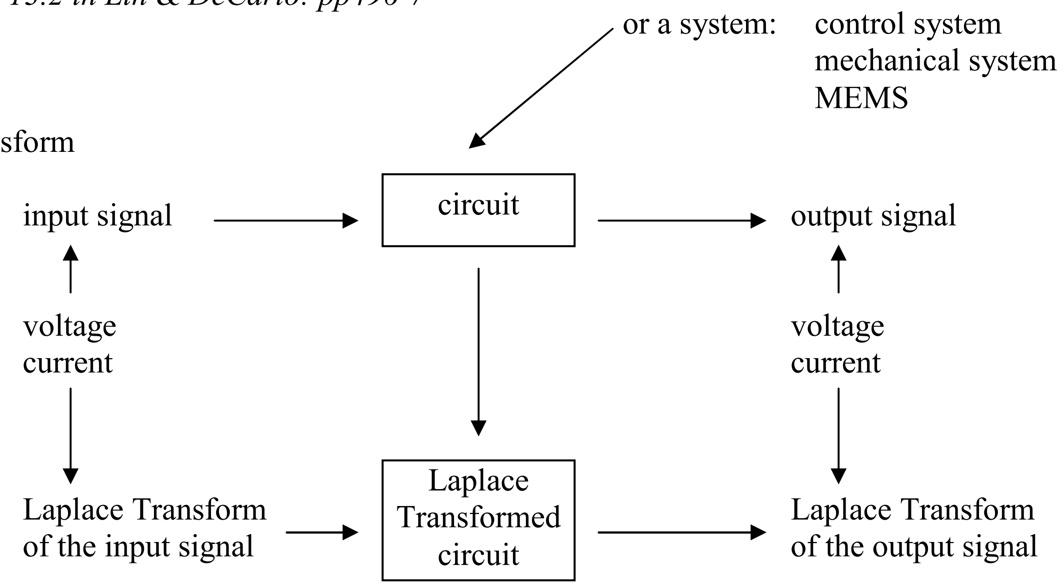
$$\left.\frac{dx_1}{dt}\right|_{0+} \qquad \left.\frac{dx_2}{dt}\right|_{0+} \qquad \text{are not known}$$

They had to be found from the DE and $x_1(0\text{-})$ and $x_2(0\text{-})$. Hence, even in this simple case, finding constants required some work and thought.

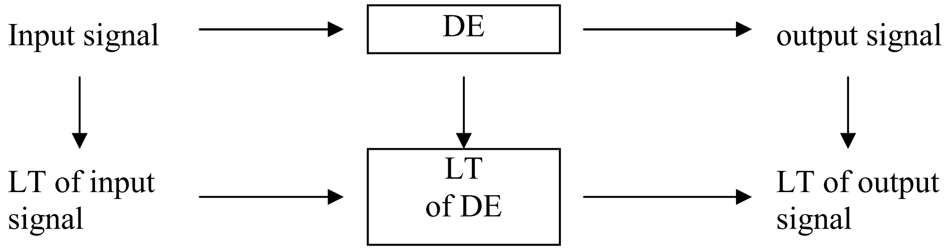
For more complicated circuits, the procedure fails because it gets rather complicated to find the constants!

See: example 13.2 in Lin & DeCarlo: pp496-7

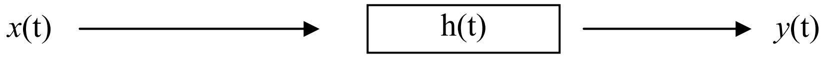
Laplace Transform



Equivalently:



Transform: a function that describes the behaviour of the system (we will learn more about it in Ch. 16 of Lin & DeCarlo)



Time domain: $y(t) = x(t) \underset{\uparrow}{*} h(t)$, where h(t): impulse response of the circuit

convolution

Laplace Transformation

$$Y(s) = H(s) \underset{\uparrow}{\cdot} X(s)$$

simple multiplication

$X(s) = L[x(t)]$
$Y(s) = L[y(t)]$
$H(s) = L[h(t)]$, where L refers to the Laplace Transform

H(s): transfer function (note: it can be found directly from the circuit after the circuit is "transformed" by the Laplace transform).

Hence,

$$x(t) \xrightarrow{L} X(s) \xrightarrow{H(s)\cdot X(s)} Y(s) \xrightarrow{L^{-1}} y(t)$$

L: Laplace Transform $\int_{0-}^{\infty} x(t)e^{-st}dt$

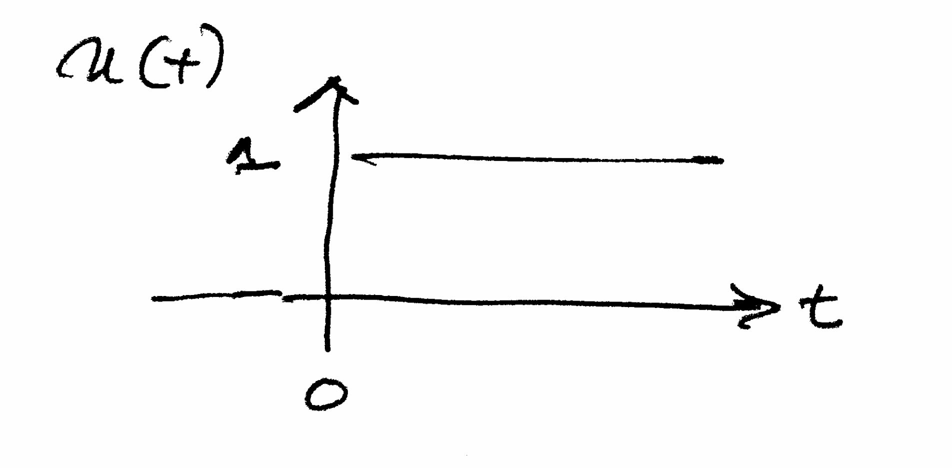
definitions

L^{-1}: Inverse Laplace Transform $\frac{1}{2\omega j}\int X(s)e^{st}ds$

Note: convolution in time domain is equivalent to multiplication in Laplace domain, which is good because convolution is difficult to apply.)

Basic signals:

$$u(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0 \end{cases}$$ unit step function



$L[u(t)] = U(s)$

By definition: $L[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$ one-sided Laplace transform, $s = \sigma + j\omega$ & $j = \sqrt{-1}$

$$U(s) = \int_{0-}^{\infty} 1 \cdot e^{-st} dt$$

$$U(s) = -\frac{1}{s} e^{-st} \Big|_{0-}^{\infty}$$

$$U(s) = \frac{1}{s}$$

$U(s) = \frac{1}{s}$: Laplace transform of u(t): unit step function

Discussion:

1. One-sided: $\int_{0-}^{\infty} f(t) e^{-st} dt$ not -∞: if we use "-∞", then we talk about two-sided L.T.

2. Why 0-?
   To account for initial conditions in a circuit. Note that we know:
   $v_c(t=0-)$
   $i_L(t=0-)$

3. What if $f(t) \neq 0$ for $t < 0$?
   In circuit analysis: usually $f(t) \equiv 0$ for $t < 0$.

4. What type of functions have Laplace Transforms?
   Not every function!

Exponentially bounded functions have Laplace Transforms (though not a necessary condition).

The integral $\int_{0-}^{\infty} f(t) e^{-st} dt$ should exist.

Consider: $f(t) = e^{t^2} u(t)$

$$F(s) = \int_{0-}^{\infty} e^{t^2} e^{-st} dt$$

$$F(s) = \int_{0-}^{\infty} e^{\left(t^2 - st\right)} dt \qquad s = \sigma + j\omega$$

$$F(s) = \int_{0-}^{\infty} e^{t^2 - \sigma t} \cdot e^{-j\omega t} dt \qquad \text{Euler constant jsin}\omega t$$

As $t \to \infty, e^{t^2 - \sigma t} \to \infty$

$\int$ = area under $e^{t^2 - \sigma t} \to \infty$

Note that this integral does not exist for any σ.

5. Why is σ important?

Consider the unit step function:

$$L[\mathrm{u(t)}] = \int_{0-}^{\infty} u(t)e^{-st}\,dt \qquad s = \sigma + j\omega$$

$$= \int_{0-}^{\infty} e^{-(\sigma+j\omega)t}\,dt$$

$$= -\frac{e^{-(\sigma+j\omega)t}}{\sigma + \mathrm{j}\omega}\Bigg|_{0-}^{\infty}$$

$$= \frac{1}{\sigma + \mathrm{j}\omega}$$

$$= \frac{1}{s}$$

But only if $\sigma > 0$

$$e^{-(\sigma+j\omega)t} = e^{-\sigma t}\cdot(\cos\omega t - j\sin\omega t)$$

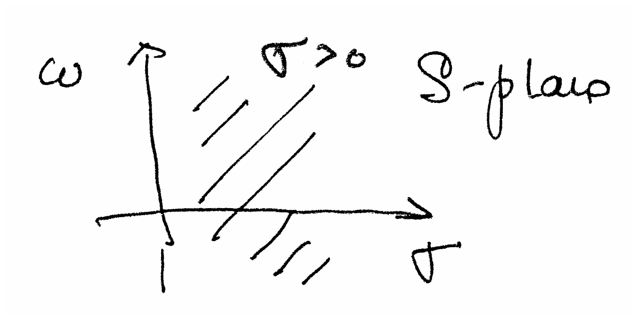
as $t \to \infty$

$$e^{-\sigma t} \to 0$$

if $\underline{\sigma > 0}$

RoC: Region of Convergence for u(t)

$\sigma > 0$



6. Is L.T. valid only within the RoC?

No. There is analytical calculation methods that permit extension to the entire S plane.

This is the reason that we usually do not mention Region of Convergence when dealing with one-sided Laplace Transforms.

Properties: L.T. is linear (nice!)

$L[a_1 f_1(t) + a_2 f_2(t)] = a_1 L[f_1(t)] + a_2 L[f_2(t)]$

- scaling with a constant
- superposition (both have linearity)

Recall: $L[u(t)] = \frac{1}{s}$

Other signals to remember: