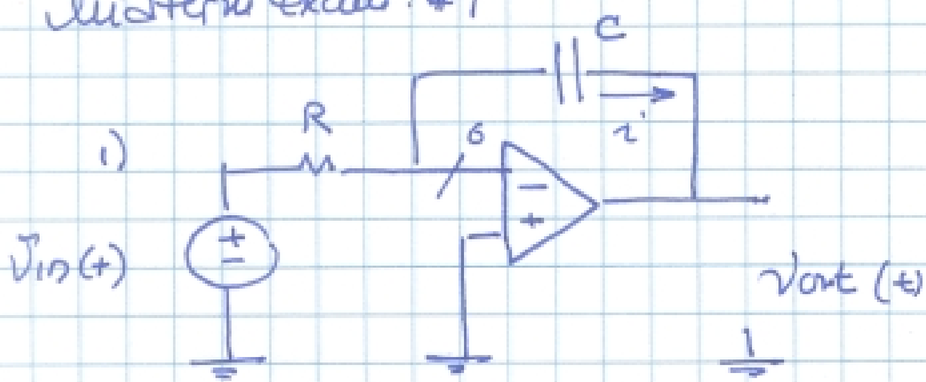
Midterm exam : #1

1)



$$v_{in} = R \cdot i$$

$$i = -C \cdot \frac{dV_{out}}{dt}$$

$$V_{in} = -RC \cdot \frac{dV_{out}}{dt}$$

$$V_{out} = -\frac{1}{RC}\int_0^t v_{in}\, dt$$

This is an integrator circuit

If $v_{in}(t) = \cos(2\pi t)$

$R = 4\,k\Omega$

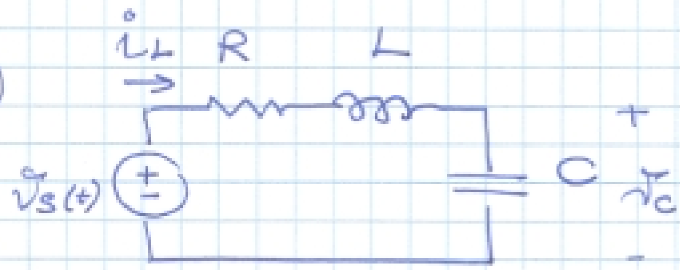
$C = 1\mu F$ $\quad RC = 4\times10^{-3}\,s$

$$V_{out} = -\frac{1}{4\times10^{-3}} \cdot \int_0^t (\cos(2\pi t))\, dt$$

$$V_{out} = -\frac{1}{4\times10^{-3}} \cdot \frac{1}{2\pi} \cdot \sin(2\pi t)$$

$$V_{out} = -\sin(2\pi t)$$

2)



$$v_s = R i_L + L\frac{di_L}{dt} + v_C$$ Notation: $v_s = v_s(t)$

$$i_L = C\frac{dv_C}{dt}$$ (Note: $i_C = i_L$)

$$v_L = L\frac{di_L}{dt}$$

Hence:

$$R \cdot i_L + L\frac{di_L}{dt} + v_C = v_s$$

$$i_L = C\frac{dv_C}{dt}$$

and:

$$\frac{di_L}{dt} = -\frac{R}{L} \cdot i_L - \frac{1}{L} v_C + \frac{1}{L} \cdot v_s$$

$$\frac{dv_C}{dt} = \frac{1}{C} \cdot i_L$$

Matrix form:

$$\frac{d}{dt} \cdot \begin{pmatrix} i_L \\ v_C \end{pmatrix} = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \begin{pmatrix} i_L \\ v_C \end{pmatrix} + \begin{pmatrix} 1/L \\ 0 \end{pmatrix} v_s$$

Natural frequence:

$$\det(sI - A) = 0$$

$$\det\left[s\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix} \right] = 0$$

$$\det\begin{pmatrix} s + R/L & 1/L \\ -1/C & s \end{pmatrix} = 0$$

$$s(s + R/L) + \frac{1}{LC} = 0$$

$$s^2 + \frac{R}{L}\cdot s + 1/LC = 0$$

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Cases:

1. $s_1 \neq s_2$ and both real

$$s_{1/2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

2. $s_1 = s_2$, real and identical

$$\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$$

i.e., $R = 2L/\sqrt{LC}$

$$R = 2\sqrt{\frac{L}{C}}$$

3. $s_1 \neq s_2$ and complex (conjugate)

$$s_{1/2} = -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

Waveforms:

1. overdamped : $s_1 \neq s_2$ and real
2. critically damped : $s_1 = s_2$
3. underdamped : $s_1 \neq s_2$ and complex.
4. undamped

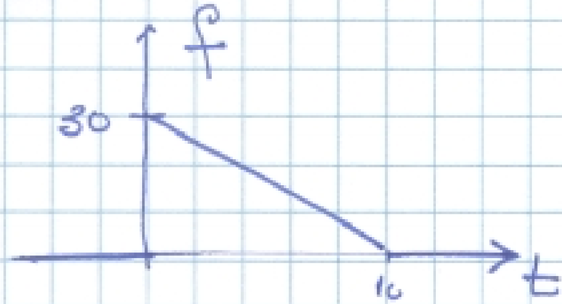
Special case: $R=0$

Ideal LC circuit with no resistance.

$s_{1/2} = \pm j \, 1/\sqrt{LC}$

undamped sustained oscillations

c)



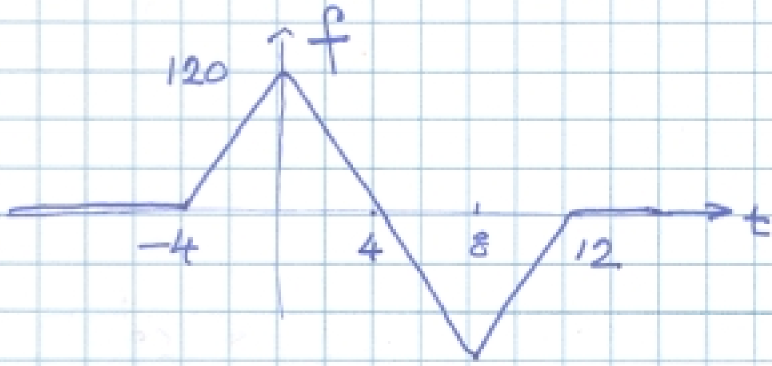
$$f(t) = -\frac{30}{10}\cdot t\cdot\left(u(t) - u(t-10)\right)$$

$$f(t) = -3t\,u(t) + 3t\,u(t-10)$$

$$f(t) = -3t\,u(t) + 3(t-10)\cdot u(t-10) + 30\,u(t-10)$$

$$F(s) = -\frac{3}{s^2} + \frac{3}{s^2}e^{-10s} + \frac{30}{s}e^{-10s}$$

d)



$$f(t) = 30\,(t+4)\cdot[u(t+4) - u(t)]$$
$$-30\,(t-4)\,[u(t) - u(t-8)]$$
$$+30\,(t-12)\,[u(t-8) - u(t-12)]$$

$$f(t) = 30\,(t+4)\,u(t+4) - 30\,(t+4)\,u(t)$$
$$-30\,(t-4)\,u(t) + 30\,(t-4)\,u(t-8)$$
$$+30\,(t-12)\,u(t-8) - 30\,(t-12)\,u(t-12)$$

$$f(t) = 30(t+4)u(t+4) - 30t\,u(t) - \cancel{120u(t)}$$
$$-30t\,u(t) + \cancel{120u(t)} + 30(t-8)u(t-8) + \cancel{120u(t-8)}$$
$$+30(t-8)u(t-8) - \cancel{120u(t-8)} - 30(t-12)u(t-12)$$

$$f(t) = 30(t+4)u(t+4) - 60t\,u(t) +$$
$$+60(t-8)u(t-8) - 30(t-12)u(t-12)$$

$$F(s) = \frac{30}{s^2}e^{4s} - \frac{60}{s^2} + \frac{60}{s^2}e^{-8s} - \frac{30}{s^2}e^{-12s}$$

4.

a. $F(s) = \dfrac{3s+1}{(s+2)(s^2+4s+8)}$

b. $F(s) = \dfrac{s(s+2)\,e^{-3s}}{(s+1)^2 \cdot (s+4)}$

a) $F(s) = \dfrac{3s+1}{(s+2)(s^2+4s+8)}$

$$= \frac{A}{s+2} + \frac{Bs+C}{s^2+4s+8}$$

$$A = \lim_{s \to -2} (s+2) \cdot F(s)$$

$$A = \lim_{s \to -2} \frac{3s+1}{s^2+4s+8}$$

$$A = \frac{-6+1}{4-8+8}$$

$$A = -\frac{5}{4}$$

Hence: $F(s) = -\dfrac{5}{4} \cdot \dfrac{1}{s+2} + \dfrac{Bs+C}{s^2+4s+8}$

Hence: $\dfrac{Bs+C}{s^2+4s+8} = \dfrac{3s+1}{(s+2)(s^2+4s+8)} + \dfrac{5}{4} \cdot \dfrac{1}{s+2}$

Note that

$$\frac{3s+1}{(s+2)(s^2+4s+8)} + \frac{5}{4(s+2)}$$

$$= \frac{4(3s+1) + 5\cdot(s^2+4s+8)}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s^2+32s+44}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s(s+2) + 22(s+2)}{4(s+2)(s^2+4s+8)}$$

$$= \frac{5s+22}{4(s^2+4s+8)}$$

$$= \frac{5s+22}{4[(s+2)^2+4]}$$

$$= \frac{5(s+2)}{4[(s+2)^2+4]} + \frac{12}{4[(s+2)^2+4]}$$

$$F(s) = -\frac{5}{4(s+2)} + \frac{5(s+2)}{4[(s+2)^2+4]} + \frac{3}{(s+2)^2+4}$$

$$f(t) = \left\{ -\frac{5}{4}e^{-2t} + \frac{5}{4}\cos 2t\, e^{-2t} + \frac{3}{2}\sin 2t\, e^{-2t} \right\} u(t)$$

$$f(t) = \frac{1}{4}\left\{ -5 + 5\cos 2t + 6\sin 2t \right\} e^{-2t} u(t)$$